

A History of Quantification

Daniel Bonevac

September 15, 2011

Aristotle (384–322 BC), the founder of the discipline of logic, also founded the study of quantification. Normally, Aristotle begins a topic by reviewing the common opinions, including the opinions of his chief predecessors. In logic, however, he could not adopt the same strategy; before him, he reports, “there was nothing at all” (*Sophistical Refutations* 183b34–36). Aristotle’s theory dominated logical approaches to quantification until the nineteenth century.

That is not to say that others did not make important contributions. Medieval logicians elaborated Aristotle’s theory, structuring it in the form familiar to us today. They also contemplated a series of problems the theory generated, devising increasingly complex theories of semantic relations to account for them. Textbook treatments of quantification in the seventeenth and nineteenth centuries made important contributions while also advancing some peculiar theories based on medieval contributions.

Modern quantification theory emerged from mathematical insights in the middle and late nineteenth century, displacing Aristotelian logic as the dominant theory of quantifiers for roughly a century. It has become common to see the history of logic as little more than a prelude to what we now call classical first-order logic, the logic of Frege, Peirce, and their successors. Aristotle’s theory of quantification is nevertheless in some respects more powerful than its modern replacement. Aristotle’s theory combines a relational conception of quantifiers with a monadic conception of terms. The modern theory combines a monadic conception of quantifiers with a relational theory of terms. Only recently have logicians combined relational conceptions of quantifiers and terms to devise a theory of generalized quantifiers capable of combining the strengths of the Aristotelian and modern approaches.

There is no theory-neutral way of defining quantification, or even of delineating the class of quantifiers. Some logicians treat determiners such as ‘all,’ ‘every,’ ‘most,’ ‘no,’ ‘some,’ and the like as quantifiers; others think of them as denoting quantifiers. Still others think of quantifiers as noun phrases containing such determiners (‘all men,’ ‘every book,’ etc.). Some include other noun phrases (‘Aristotle,’ ‘Peter, Paul, and John,’ etc.). Some define quantifiers as variable-binding expressions; others lack the concept of a variable. My sketch of the history of our understanding of quantification thus traces the development of understandings of *what* is to be explained as much as *how* it is to be explained.

1 Aristotle's Quantification Theory

Aristotle first developed a theory of quantification in the form of his well-known theory of syllogisms. The theory's familiarity, not only from ubiquitous textbook treatments but also from important scholarly studies, should not blind us to some of its less-remarked but critically important features.¹ Aristotle recognizes that validity is a matter of *form*. He aspires to *completeness*; he characterizes a realm of inquiry and seeks to identify all valid argument forms within it. He develops the first theory of *deduction*, and offers the first *completeness proof*, showing by means of his method of deduction that all the valid argument forms within that realm can be shown to be valid on the basis of two basic argument forms. He proceeds to prove several *metatheorems*, which taken together constitute an alternative decision procedure for arguments. More importantly for our purposes, Aristotle develops an understanding of quantifiers that is in some ways more powerful than that of modern logic, and was not superseded until the development of the theory of generalized quantifiers.

1.1 Validity as a Matter of Form

Aristotle restricts his attention to *statements, assertions, or propositions (apophanseis)*, sentences that are (or perhaps can be) true or false.² Like most subsequent logicians, he focuses on a limited set of quantifiers. Every premise, he says, is universal, particular, or indefinite:

A proposition (*protasis*), then, is a sentence affirming or denying something of something; and this is either universal or particular or indefinite. By universal I mean a statement that something belongs to all or none of something; by particular that it belongs to some or not to some or not to all; by indefinite that it does or does not belong, without any mark of being universal or particular, e.g. 'contraries are subjects of the same science', or 'pleasure is not good'. (*Prior Analytics* I, 1, 24a16–21.)³

Aristotle is here defining the subject matter of his theory; interpreted otherwise, his claim is obviously false. Not only are some statements singular, e.g., 'Socrates is running,' as Aristotle recognizes at *De Interpretatione* I, 7, while others are compound, but there are also many other quantifiers: 'many,' 'most,' 'few,' 'exactly one,' 'almost all,' 'finitely many,' and so on. The theory he presents, in fact, focuses solely on universal and particular quantifiers; he has almost nothing to say about the third sort he mentions, indefinites (or bare plurals, as they are known today), and the little he says holds only in limited linguistic contexts.⁴

¹Important scholarly studies include Kneale and Kneale 1962, Patzig 1969, Corcoran 1972, 1973, 1974, Smiley 1974, 1994, Lear 1980, Barnes 1981, Irwin 1988, Wedin 1990, McKirahan 1992, Johnson 1994, Boger 2004, and Woods and Irvine 2004.

²See *De Interpretatione* 4, 17a1: "Every sentence is significant... but not every sentence is a statement-making sentence, but only those in which there is truth and falsity."

³Hereafter all references to Aristotle will be to the *Prior Analytics* unless otherwise noted.

⁴"We shall have the same deduction (*syllogismos*) whether it is indefinite or particular" (I, 4, 26a30; see also I, 7, 29a27–29: "It is evident also that the substitution of an indefinite for a particular affirmative

The sentences within the scope of Aristotle's theory, then, are either universal or particular. They are also either affirmative or negative (I, 2, 25a1). (Medieval logicians refer to these as the *quantity* and *quality* of the statements, respectively. Peter of Spain may have been the first to use those terms. C. S. Peirce credits their introduction to Apuleius (125?–180?), remarking that they are “more assified than golden” (1893, 279n), but this rests on a misattribution.) Aristotle thus concerns himself with four kinds of sentences, known as *categorical propositions*. These have the forms:

Universal affirmative (A): Every S is P (SaP)

Universal negative (E): No S is P (SeP)

Particular affirmative (I): Some S is P (SiP)

Particular Negative (O): Some S is not P (SoP)

I shall call these *categorical statement forms*. The abbreviations A, E, I, and O are early medieval inventions; the symbolic forms SaP , etc., are not in Aristotle or the medievals, though a number of modern commentators employ them. They point to two central features of Aristotle's theory. First, Aristotle analyzes inference in terms of logical form. He does not turn to a discussion of the content of the terms that appear in the arguments he uses as illustrations (e.g., ‘contraries,’ ‘pleasure,’ ‘good,’ and the like, from the arguments in the very first section). He sees logical validity as structural, as a matter of form. He introduces the use of variables to represent the forms of the sentences under consideration, leaving only the determiners, the copula, and negation as logical constants. In medieval language, he treats them as *syncategorematic*. He proceeds to develop a theory of validity, assuming that telling good arguments from bad is solely a matter of identifying and classifying logical forms.

Second, the notation SaP , etc., makes explicit Aristotle's understanding of quantifiers as relations. It is not so easy to say what they relate; given Aristotle's lack of any theory of the semantic value of terms (*horoi*), it is probably most accurate to say that, for him, quantifiers are relations between terms.⁵ This marks a critical difference between Aristotle's theory and modern quantification theory, which effectively takes a quantifier as a monadic predicate.

Aristotle uses the singular (e.g., ‘Every S is P ’) rather than the plural, which sounds more natural in English, to stress that terms such as S and P are affirmed or denied of objects taken one by one. His definition of ‘term’ is obscure—“I call that a term into which the proposition is resolved, i.e. both the predicate and that of which it is predicated, ‘is’ or ‘is not’ being added” (I, 1, 24b17–18)—but it is plainly in the general spirit of his theory to define a term as a linguistic expression that can be true or false of individual objects. He does not put it in quite that way, for he thinks of truth

will effect the same deduction in all the figures.”). Aristotle's commentators follow him in this. Alexander of Aphrodisias, for example, writes: “He [Aristotle] doesn't speak about [converting] indefinites, because they are of no use for syllogisms and because they can be [regarded as] equal to particulars” (Alexander of Aphrodisias 1883, 30.) ‘Dogs are barking’ is equivalent to ‘Some dogs are barking,’ but ‘Dogs bark’ is plainly not equivalent to ‘Some dogs bark.’ The position thus has some peculiar consequences. There are no valid syllogisms with two indefinite premises, for example; hence, ‘Dogs are mammals; mammals are animals; so, dogs are animals’ and ‘Dogs are mammals; dogs bark; so, some mammals bark’ both fail.

⁵For a discussion of extensional and intensional interpretations of the role of quantifiers, see Boger (2004), 164–165.

and falsehood solely as properties of statements; he has no concept of an expression being true or false *of* something, though he does have the closely related concept of an expression being affirmed or denied of something.⁶

The determiner-first phrasing ('Every *S* is *P*,' 'No *S* is *P*,' etc.) is from Boethius (480–525). Aristotle's usual phrasing places the determiner near the end: '*P* belongs to every *S*,' '*P* belongs to no *S*,' etc.), which perhaps makes it easier to see the validity of some syllogistic forms, as certain commentators have suggested, but also makes it harder to see the determiner as a relation between terms. To a modern ear, it furthermore suggests a misleading analogy with set theory. I will consequently continue to use the Boethian representations, even though they post-date Aristotle by some 800 years.

1.2 Aristotle's Completeness Proof

Aristotle has an informal definition of validity:

A syllogism is discourse in which, certain things being stated, something other than what is stated follows of necessity from their being so. I mean by the last phrase that they produce the consequence, and by this, that no further term is required from without in order to make the consequence necessary. (I, 1)

Notice that syllogisms are thus valid by definition; most arguments of the general sort addressed by the theory are not syllogisms. Notice also that the definition has a circular feel to it, for the definiens contains 'follows,' and its explication contains 'consequence.' Aristotle thus says that a syllogism is one whose premises entail its conclusion. But he does not give even an informal characterization of entailment. Some commentators read the contemporary conception of deductive validity as truth-preservation—if the premises are true, the conclusion must be true as well—into Aristotle's definition on the ground that he speaks of the consequence following by necessity (*ex anankes sumbainein*). It seems fairer to treat the modern conception as compatible with Aristotle's definition, but as not itself a part of the theory, since Aristotle's definition makes no reference to truth. He seems to take the concept of necessity as primitive (Lear 1980). Aristotle also insists that the conclusion must be a proposition that does not appear among the premises. Some ancient commentators, taking Aristotle's plural form seriously—in my view, too seriously—furthermore see the definition as ruling out arguments with a single premise. In any case Aristotle does concern himself with such inferences under the heading of conversion.

It has become common to see in Aristotle not only the first development of a logical system but also the first completeness proof. In one sense, this is correct, as we shall see in a moment. But it is also misleading, for Aristotle has nothing more than an intuitive conception of validity. He restricts himself to arguments with two premises, one conclusion, and three terms, each appearing in two propositions. All the propositions are of one of four categorical statement forms. That makes his task finite. There are 256 possible arguments of that general form, 24 of which he considers valid. He

⁶Compare *De Interpretatione* 10, 19b11–12: "Without a verb there will be no affirmation or negation."

then shows that all the valid forms reduce to some basic forms in the sense that their validity can be demonstrated from the validity of the basic forms by applying certain rules. From a modern perspective, the finite character of the deduction system makes it something of a toy. One could simply take all twenty-four valid forms as basic and leave it at that. The real point of axiomatization is, after all, to give a finite characterization of an infinite set, something Aristotle has no need to do within the confines of the theory of categorical syllogisms. The interesting completeness claim lies not in the system of the *Prior Analytics* but instead in the *Posterior Analytics*, where Aristotle argues that all deductively valid arguments can be shown to be valid by combinations of syllogistic techniques.⁷

Aristotle's system is nevertheless an impressive achievement. He effectively presents something akin to a natural deduction system for syllogistic forms. He presents the rules as licensing moves from one categorical sentence form to another, and most commentators follow him in thinking of the deduction system as licensing inferences to categorical statement forms given other categorical statement forms. Lukasiewicz (1951), noting that Aristotle always phrases syllogisms as conditionals rather than inference patterns, constructs an Aristotelian calculus with some interesting features. Given that Aristotle's conditionals are logical truths, however, and given that Lukasiewicz construes them as material conditionals, the deduction theorem makes a conditional interpretation equivalent to an inference rule interpretation.⁸ It would also be possible to interpret the derivations Aristotle provides in the style of a Gentzen consecution calculus, as licensing moves from one syllogism to another. I shall present his system in both forms.

Aristotle distinguishes three syllogistic *figures*, or configurations of terms. Syllogisms contain three categorical propositions and three terms, each of which appears in two different propositions. The *middle* term occurs in both premises; the other terms are *extremes*. In the *first* figure, as Aristotle understands it, the middle term is subject of one premise and predicate of the other. (This, at any rate, is the definition offered by Theophrastus, Aristotle's successor as head of the Lyceum; Aristotle himself gives no definition.⁹) In the *second* figure, the middle term appears as predicate in both premises (26b34–35). In the *third* figure, the middle term appears as subject in both (28a10–11).

Aristotle's commentators, as well as many medieval logicians, debate the need for a fourth figure. The issue comes down to distinguishing the extremes as major and minor terms. Aristotle gives no general definition; instead, he distinguishes major from minor terms relative to a figure. Here are his definitions:

Major term

First figure: "in which the middle is contained" (26a22)

Second figure: "that which lies near the middle" (26b37)

⁷For an extended discussion, see Lear (1980).

⁸There are reasons other than tradition and this equivalence for taking syllogisms as arguments rather than conditionals. Lukasiewicz has to attribute to Aristotle a modern propositional logic; he also has to reinterpret talk of premises and conclusions as talk of antecedents and consequents.

⁹See Lukasiewicz (1951), 97. Some commentators see a definition in Aristotle, at 25b32–33 or 40b30–41a20. I find it hard to interpret these remarks as definitions, much less as a definition specifying that the middle term is subject of the major premise and predicate of the minor premise.

Third figure: “that which is further from the middle” (28a13)

Minor term

First figure: “which comes under the middle” (26a23)

Second figure: “that which is further away from the middle” (26b38)

Third figure: “that which is nearer to [the middle]” (28a14)

These are unsatisfactory in several respects. They suggest no general meaning of ‘major’ and ‘minor’; the two switch roles completely between the second and third figures. The definition for first figure works well for the syllogism known to the medievals as *Barbara*:

Every M is P
Every S is M
 \therefore Every S is P

But it fails for others. Consider *Ferio*:

No M is P
Some S is M
 \therefore Some S is not P

Here the middle term, M , is not contained in P ; in fact, the two are completely disjoint.

John Philoponus (490–570), a sixth-century Alexandrian commentator, first defines major and minor terms in the way that is now standard: The major term is the predicate of the conclusion; the minor term is the subject of the conclusion.¹⁰ That definition, of course, makes it easy to distinguish two figures within what Aristotle considered the first: those in which the middle term is the subject of the major premise (which remain first figure) and those in which it is subject of the minor premise (fourth figure). It is then possible to specify a syllogism completely by indicating its figure (first, second, third, or fourth) and its mood (the categorical statement forms in the order ⟨major premise, minor premise, conclusion⟩): thus, 1AAA, 2EIO, 3IAI, etc.

Aristotle’s deduction system proceeds by listing acceptable immediate inferences, inferences from one categorical statement form to another, that act as rules of inference. The first and most important is *conversion*. Particular affirmative and universal negative statement forms convert:

Conversion

Some S is $P \Leftrightarrow$ Some P is S
No S is $P \Leftrightarrow$ No P is S

¹⁰In *Aristotelis Analytica Priora commentaria*, Wallies, ed., 67.27–29, quoted in Spade (2002), 20: “So we should use the following rule for the three figures, that the major is the term in predicate position in the conclusion, and the minor [is the term] in subject position in the conclusion.”

A second inference rule is *conversion per accidens*, and raises issues of existential import, to which we will turn in the next section. Universal affirmatives do not convert *simply*, as in the above rule; ‘every S is P ’ and ‘every P is S ’ are not equivalent. But Aristotle assumes that all terms have nonempty extensions, and so allows the move from A forms to I forms:

Conversion per accidens

Every S is $P \Rightarrow$ Some P is S

Aristotle’s third inference rule is *reductio ad absurdum*, or indirect proof, together with rules about which categorical statement forms contradict which:

Contradictories

NOT Every S is $P \Leftrightarrow$ Some S is not P

NOT Some S is not $P \Leftrightarrow$ Every S is P

NOT Some S is $P \Leftrightarrow$ No S is P

NOT No S is $P \Leftrightarrow$ Some S is P

Thus, A and O forms are contradictories, as are I and E forms; they always have opposite truth values.

Armed with these rules, we can reduce some syllogisms to others. Aristotle first shows that we can reduce all syllogisms to first-figure patterns. The first-figure syllogisms serve as axioms, if we think of the deduction system as moving us from syllogisms to syllogisms in the style of a consecution calculus, or as inference rules, if we think of it as moving us from statement forms to statement forms in a more typical natural deduction system. From first-figure syllogisms we can deduce all syllogisms by means of the other rules of inference.

Most deductions are direct. Consider, for example, the second figure pattern known as *Cesare* (2EAE):

No P is M
 Every S is M
 \therefore No S is P

We can convert the first premise to obtain the first-figure syllogism *Celarent* (1EAE):

No M is P
 Every S is M
 \therefore No S is P

We can think of this deduction in two ways. The first is a series of steps, each of which is a categorical statement form:

1. No P is M (Assumption)
2. Every S is M (Assumption)

3. No M is P (Conversion, 1)

4. No S is P (1EAE, 3, 2)

Note that we here use a first-figure syllogism as a rule of inference.

The second, in the style of the Gentzen consecution calculus, is a quick deduction from one syllogism to another:

1. No M is P , Every S is M \vdash No S is P (1EAE)

2. No P is M , Every S is M \vdash No S is P (Conversion, 1)

Which style of deduction most closely matches Aristotle's conception? Probably the former. Here is his deduction of *Cesare* (substituting variables appropriately):

Let M be predicated of no P , but of every S . Since, then, the negative is convertible, P will belong to no M : but M was assumed to belong to every S : consequently P will belong to no S . This has already been proved.

This seems to take the lines of the proof as categorical statement forms rather than syllogisms.

Two syllogisms cannot be reduced to first-figure syllogisms directly. Consider this second-figure syllogism, *Baroco* (2A00):

Every P is M
Some S is not M
 \therefore Some S is not P

We can pursue an indirect proof:

1. Every P is M (Assumption)
2. Some S is not M (Assumption)
3. NOT Some S is not P (Assumption for reductio)
4. Every S is P (Contradictories, 3)
5. Every S is M (1AAA, 1, 4)
6. NOT Every S is M (Contradictories, 2)
7. Some S is not P (Reductio, 3–6)

If we think of the system as moving from syllogisms to syllogisms, we can frame the argument in this way, where we view reductio as a rule allowing us to move from a syllogism $p, q \vdash r$ to $p, \text{NOT } r \vdash \text{NOT } q$ (or $q, \text{NOT } r \vdash \text{NOT } p$), and then switch the order of the premises:

1. Every P is M , Every S is P \vdash Every S is M (1AAA)
2. NOT Every S is M , Every P is M \vdash NOT Every S is P (Reductio, 1)

3. Some S is not M , Every P is M \vdash Some S is not P (Contradictories, 2)

4. Every P is M , Some S is not M \vdash Some S is not P (Order, 3)

Aristotle can show, in either fashion, that all syllogisms reduce to first-figure syllogisms.

In fact, they all reduce to just two patterns, 1AAA (*Barbara*), which we have already met, and 1AII (*Darii*):

Every M is P
Some S is M
 \therefore Some S is P

For the moment, consider Aristotle's more restricted claim. All we need to do to show this is reduce the other first-figure patterns to those two. To take an example, consider the first-figure syllogism *Ferio* (1EIO):

No M is P
Some S is M
 \therefore Some S is not P

We can reduce this as follows:

1. No M is P (Assumption)
2. Some S is M (Assumption)
3. NOT Some S is not P (Assumption for reductio)
4. Every S is P (Contradictories, 3)
5. Some M is S (Conversion, 2)
6. Some M is P (1AII, 4, 5)
7. NOT Some M is P (Contradictories, 1)
8. Some S is not P (Reductio, 3–7)

Or, in consecution form,

1. Every S is P , Some M is S \vdash Some M is P (1AII)
2. Every S is P , Some S is M \vdash Some M is P (Conversion, 1)
3. NOT Some M is P , Some S is M \vdash NOT Every S is P (Reductio, 2)
4. No M is P , Some S is M \vdash Some S is not P (Contradictories, 3)

Aristotle does not take *Darii* as fundamental. Instead, he argues that all syllogisms reduce to universal first-figure syllogisms. He shows, for example, how *Darii* reduces by reducing it to a second-figure syllogism, which in turn reduces to first-figure. His argument (again, replacing variables):

... if P belongs to every M , and M to some S , it follows that P belongs to some S . For if it belonged to no S , and belongs to every M , then M will belong to no S : this we know by means of the second figure. (29b8–10)

We can represent this as follows:

1. Every M is P (Assumption)
2. Some S is M (Assumption)
3. NOT Some S is P (Assumption for reductio)
4. No S is P (Contradictories, 3)
5. No S is M (2AEE, 1, 4)
6. NOT No S is M (Contradictories, 2)

This reduces *Darii* to the second-figure syllogism *Camestres*, which in turn reduces to the universal first-figure syllogism *Celarent*:

1. Every M is P (Assumption)
2. No S is P
3. No P is S (Conversion, 2)
4. No M is S (1EAE, 3, 1)
5. No S is M (Conversion, 4)

But we could also reduce universal negative first-figure syllogisms to *Darii*. So, it is possible to see every syllogism as derived rule of inference if we take *Barbara* and either *Celarent* or *Darii* as basic rules.

1.3 The Square of Opposition

In *De Interpretatione* 7 Aristotle discusses logical relations between categorical propositions, formulating what has become known as the square of opposition. As early as the second century CE, diagrams of the square began to appear in logic texts; they became commonplace by the twelfth century. As we have seen, Aristotle takes universal affirmatives and particular negatives as contradictories, propositions having opposite truth values. He similarly takes particular affirmatives to contradict universal negatives. Aristotle adds to these relations several others.

First, and explicitly, universal affirmatives and universal negatives are *contraries*: they cannot both be true. ‘Every S is P ’ and ‘No S is P ’ may both be false, but cannot both be true (17b3–4, 21).

Second, in consequence, particular affirmatives and negatives are what later came to be known as *subcontraries*: they can both be true, but cannot both be false. Suppose ‘Some S is P ’ and ‘Some S is not P ’ were both false. Then their contradictories, ‘Every S is P ’ and ‘No S is P ,’ would both be true. As contraries, however, they cannot be.

Third, universals imply their corresponding particulars. Suppose ‘Every S is P ’ is true. Then its contrary ‘No S is P ’ is false, so its contradictory ‘Some S is P ’ must be true. A similar argument shows that ‘No S is P ’ implies ‘Some S is not P .’

Aristotle’s theses about contraries and contradictories thus generate a problem. ‘Some S is P ’ and ‘Some S is not P ’ cannot both be false, so ‘Some S is P or some S is not P ’ is evidently a logical truth. But it appears to be equivalent to ‘Some S is either P or not P ,’ and that appears to be equivalent to ‘There is an S .’ So, Aristotle’s theses imply that certain existence claims are logical truths.

We can put the point simply. Suppose there are no S s. Then evidently ‘Some S is P ’ and ‘Some S is not P ’ are both false, in which case their contradictories, ‘Every S is P ’ and ‘No S is P ,’ must both be true. But then they cannot be contraries.

If we are to maintain the core Aristotelian claims that generate the square of opposition, there are only two options. One is to restrict the logic to nonempty terms.¹¹ The other is to insist that ‘Some S is not P ’—or ‘Not every S is P ,’ as Aristotle phrases the particular negative form in this context—can be true even though there are no S s. That strategy in turn divides into two: to treat the two as equivalent, and use the latter to motivate the thought that ‘Some S is not P ’ does not imply that there are S s, or to deny the equivalence of ‘Some S is not P ’ and ‘Not every S is P ’ as Abelard (1079–1142) does, holding that only the former has existential import and that the latter is the proper inhabitant of the square of opposition. It is interesting that Aristotle used that form when articulating the square. Boethius substitutes the usual particular negative form, however, and most logicians followed his lead, even once Abelard pointed out the possibility of drawing a distinction.

2 Quantifiers in Medieval Logic

2.1 The Old Logic

The disappearance of Aristotle’s logical works after the sixth century, when John Philoponus (490–570) and Simplicius (490–560) had access to them and wrote their important commentaries, placed medieval logicians at a great disadvantage. They knew logic chiefly through the works of Porphyry and Boethius. The logical tradition that grew in that somewhat thin soil became known as the *logica vetus*—the Old Logic.

Porphyry (234–305), a Phoenician neo-Platonist, wrote commentaries on Aristotle’s *Categories* that are now mostly lost. But he also wrote the *Isagoge* (Introduction), which became hugely influential, largely by way of Boethius’s commentary on it, throughout the following millennium. Porphyry sought to avoid deep philosophical questions, but his formulation of them influenced medieval thinkers for centuries.

¹¹Thus, Buridan (2001): “Aristotle did not intend to speak about fictive terms, namely, ones that supposit for nothing, such as ‘chimera’ or ‘goatstag,’ but about terms each of which supposit for something” (382). Inspired by the square of opposition for modal propositions, which modern logic generally still endorses (‘It is necessary that p ’ and ‘It is necessary that not p ’ are contraries, ‘It is necessary that p ’ and ‘It is possible that not p ’ are contradictories, etc.), one might think of construing Aristotle’s quantifiers as ranging over possible objects. The problem of empty terms then becomes a problem of impossible terms; what is one to say about ‘Some round squares are not square’?

Boethius (475?–526?), son of a Roman consul and father of two others, translated and commented on Porphyry’s *Isagoge* as well as Aristotle’s *Prior Analytics*. He wrote two works on categorical syllogisms. Though most scholars see him as making few original contributions to logic, and in particular to the theory of quantification, we have already seen one vital contribution, namely, the reformulation of categorical statement forms to put determiners in initial position, e.g., of ‘*P* belongs to every *S*’ as ‘Every *S* is *P*.’ That reformulation made it possible for medieval thinkers to see quantifiers straightforwardly as relations between terms, enabling them to develop the doctrine of distribution, the *dictum de omni et nullo*, and the general theory of terms that transformed the Old Logic first into the *logica nova* and then into the late medieval theory of terms.¹² Boethius also discusses conversion at length, adding to Aristotle’s theory in several respects. He has a great interest in the infinite (*infinitum*) terms discussed briefly by Aristotle in *De Interpretatione* 10, and in particular in the equivalence of ‘No *S* is *P*’ and ‘Every *S* is *nonP*’ (20a20–21), called *equipollence* throughout the medieval period and later known as *obversion*. This leads him to supplement Aristotle’s theory of categorical syllogisms in various ways. First, he adds a new form of conversion *per accidens*, allowing the transition from universal to particular negatives:

Boethian conversion per accidens

No *S* is *P* \Rightarrow Some *nonP* is *S*

This follows directly from Aristotelian conversion *per accidens* given the above equivalence. It also follows, of course, from subalternation and simple conversion. Second, Boethius adds a thoroughly new form of conversion, *contraposition*.

Discussions of contraposition are common in Old Logic texts. Illustrative is the *Abbreviatio Montana*, a short summary of logic written in the mid-twelfth century by the monks of Mont Ste. Geneviève. It provides an overview of the Old Logic, the theory that had developed from Porphyry and Boethius without the direct influence of the *Prior Analytics*. Though it follows Aristotle in most respects, it diverges from the *Prior Analytics* in some striking ways.

The first divergence concerns the nature of logic itself. Aristotle’s subject is demonstration: what follows from what. The *Abbreviatio Montana*, in contrast, locates logic within the sphere of rhetoric. The art with which it is concerned is dialectic, whose purpose is “to prove in the basis of readily believable arguments a question that has been proposed” (77). Its goal is persuasion, “to produce belief regarding the proposed question.”

A second divergence concerns the nature of quantifiers. The *Abbreviatio Montana* includes singular propositions along with universals, particulars, and indefinites in its catalogue of categorical propositions, and defines them in an interesting way, distinguishing the determiner from the noun phrase. A universal proposition is one that “has

¹²See Martin 2009. I am setting aside here another important contribution with some connection to our subject, namely, Boethius’s contribution to the theory of topics, which eventually blended into the theory of quantification in the *logica nova*. Boethius took two very different approaches to the theory of topics—those of Aristotle and Cicero (106–43 BC)—and unified them within a generally Aristotelian framework, organizing what appears in the *Topics* of both Aristotle and Cicero to be a disorganized collection of insights and putting the theory into a form in which it remained influential until the Renaissance.

a universal subject, stated with a universal sign” (80). A particular proposition, similarly, has a particular subject stated with a particular sign. An indefinite has no sign at all. A singular proposition “has a singular subject stated with a singular sign, as in ‘Socrates is reading.’” A proper name is thus both a subject and a sign.

The *Abbreviatio Montana* follows Boethius, distinguishing *finite* terms such as ‘man,’ ‘stone,’ and so on from *infinite* terms such as ‘non-man,’ ‘non-stone,’ and so on. In contraposition,

The predicate is turned into a subject, and the subject is turned into a predicate; the finite terms are made infinite, while the signs remain the same. Universal affirmatives and particular negatives are converted with this sort of conversion. How? In this way: ‘Every man is an animal’: ‘Every non-animal is a non-man’; ‘Some man is not an animal’: ‘Some non-animal is [not] a non-man’; ‘No man is an animal’: ‘No non-animal is a non-man’; ‘Some man is an animal’: ‘Some non-animal is a non-man.’ (*Abbreviatio Montana* 83)

Following Boethius, the Old Logic texts define contraposition accordingly:

Contraposition

Every S is $P \Leftrightarrow$ Every $nonP$ is $nonS$
Some S is not $P \Leftrightarrow$ Some $nonP$ is not $nonS$

The *Abbreviatio Montana* lists contraposition among the conversions applicable to propositions sharing both terms. But Aristotle would hardly count ‘Every man is an animal’ and ‘Every non-animal is a non-man’ as sharing terms. Allowing infinite terms into Aristotle’s framework, moreover, threatens to break down the entire system. It allows the expression of categorical propositions such as ‘No non-animal is a non-man’ and ‘Some non-animal is a non-man’ which have no categorical equivalents expressible in finite terms.¹³ The *Introductiones Norimbergenses* provides a table including such propositions (309, f. 54r):

Every man is an animal \Leftrightarrow No man is a non-animal
Every man is a non-animal \Leftrightarrow No man is an animal
Every non-man is an animal \Leftrightarrow No non-man is a non-animal
Every non-man is a non-animal \Leftrightarrow No non-man is an animal

The presence of infinite terms also allows the formulation of valid inference patterns that have no place in Aristotle’s system, such as

No $nonM$ is $nonP$
No S is M
 \therefore Every S is P

Such inference patterns went unnoticed in the Old Logic, and even in the New. It would be two hundred years before John Buridan would expand the theory of the syllogism to include them, and more than seven hundred years before Lewis Carroll (1896,

¹³The former is equivalent to ‘Everything is an animal or a man,’ the latter to its negation.

1977) would write about them and use them to mock the syllogistic rules of the New Logicians, which nineteenth-century textbook writers would regurgitate—Buridan’s contributions long having been forgotten. The syllogism above, for example, has several shocking features. It appears to equivocate on its middle term. It derives an affirmative conclusion from two negative premises. It is first-figure, with a universal affirmative conclusion, but it certainly is not *Barbara*.

It does, however, reduce readily to *Barbara*, given a rule of Obversion recognizing the equivalences:

Some S is $nonP \Leftrightarrow$ Some S is not P
 Every S is $nonP \Leftrightarrow$ No S is P
 No S is $nonP \Leftrightarrow$ Every S is P
 Some S is not $nonP \Leftrightarrow$ Some S is P

The deduction goes as follows:

1. No $nonM$ is $nonP$ (Assumption)
2. No S is M (Assumption)
3. Every $nonM$ is P (Obversion, 1)
4. Every S is $nonM$ (Obversion, 2)
5. Every S is P (*Barbara*, 3, 4)

It is somewhat surprising, then, that no one noticed the additional power that recognition of infinite terms and immediate inferences of contraposition and obversion grant. A defender of the Old Logic who realized its power might have been in an excellent position to combat the New Logic’s shift in focus.

Another manifestation of the added power of the Old Logic lies in its effect on Aristotle’s reduction of syllogisms to first-figure forms. Aristotle shows that all valid syllogisms reduce to *Barbara* and either *Darii* or a negative form such as *Celarent*. We could go even further with obversion, reducing *Celarent* and *Darii* to *Barbara*, thus deriving every syllogism from the single form IAAA.¹⁴

The additional power of the Old Logic, however, creates additional problems with the square of opposition. We saw earlier that if S is empty, ‘Some S is P ’ and ‘Some S is not P ’ are both false. By contradictories, then, ‘Every S is P ’ and ‘No S is P ’ must both be true. But then they cannot be contraries. That is bad enough, perhaps, though we might reassure ourselves that Aristotle intended his theory as a tool for demonstrative science, the terms of which would not be empty. Adding obversion and contraposition, however, undermine any such reassurance. The *Abbreviatio Montana* passes over the difficulties contraposition introduces, but the monks of Mount Ste. Geneviève discuss them at length in their accompanying twelfth-century manuscript *Introductiones Montana Minores*. They worry that contraposition entails not only that

¹⁴Surprisingly, no Old Logician seems to recognize this. But the deduction is not difficult. Assume *Celarent*: No M is P , Every S is $M \vdash$ No S is P . Two applications of Obversion reduce this to *Barbara*: Every M is $nonP$, Every S is $M \vdash$ Every S is $nonP$.

the subject term is nonempty, but that *no term in the language* is empty. Here, for example, is a simple argument from a completely unrelated categorical proposition to the existence of stones:

If ‘every man is an animal’ is true, so is ‘every non-animal is a non-man,’ by the same reasoning [i.e., by contraposition]. And if every non-animal is a non-man, every stone is a non-man, by the topic “from the whole” [i.e., since every stone is a non-animal]. And if every stone is a non-man, every stone exists (“from the part”). (34, f. 41rb, my translation)

Not only may there be no empty terms, whether involved in the reasoning under consideration or not; there can be no universally applicable terms.

‘Every man is a thing.’ This proposition is true; therefore its converse by contraposition, ‘every non-thing is not a man,’ is too. But if that is so, then every non-thing exists, because whatever is a non-man exists.

This shows that it is not possible to convert propositions having terms applying to everything. (34–35, f. 160r, my translation)

The monks, following Abelard, conclude that the particle ‘not’ should not be used to form infinite terms, but instead be restricted to negating propositions.

The monks also consider objections to simple conversion, but reject them as relying on misleading linguistic forms. ‘Some old man was once a boy’ might appear, outrageously, to imply that some boy was once an old man, but this is a fallacy based on tense.

2.2 The New Logic

No one knows how the logical works of Aristotle that had been missing for centuries were once again found, but increased travel to Constantinople, due in part to economic growth and in part to the Crusades, seems to have led a number of scholars to investigate Byzantine libraries and begin translation projects that included the works of Aristotle. The discovery of Aristotle’s logical works in the late twelfth century changed logic significantly, leading to the development of the *logica nova*—the New Logic.

The texts of the New Logic begin just as those of the Old Logic do, with an account of sounds, words, nouns, verbs, and the standard Aristotelian definition of a statement or proposition as an expression signifying what is true or false (*Propositio est oratio verum vel falsum significans*). But the remainder of the texts diverge significantly from those of the Old Logic in emphasis. They also include a number of new theoretical developments, including the doctrine of distribution, the development of rules for determining the validity of syllogisms, and the theory of supposition.

One sign of the shift in emphasis is the emergence of mnemonics for syllogisms and deductions. The *Ars Emmerana* and *Ars Burana*, twelfth-century Old Logic texts, end their treatments of syllogisms with some syllables that stand for the valid moods in various syllogistic figures.¹⁵ Those syllables did not catch on. But by the time of Peter

¹⁵See *Ars Emmerana* 173, 49ra; *Ars Burana* 200, f. 111r, 203, 112r, and 205, 112v. The syllables, VIO

of Spain's *Summulae Logicales* a century later, a verse encoding such information had caught on and become standard. As Peter has it:

Barbara Celarent Darii Ferio Baralipon
Celantes Dabitis Fapesmo Frisesomorum
Cesare Cambestres Festino Barocho Darapti
Felapto Disamis Datisi Bocardo Ferison.¹⁶

These expressions, seemingly nonsensical, encode a great deal of information. The first two lines are the first-figure syllogisms in Aristotle's scheme. (The second line would later be reckoned as fourth figure.) The third line, except for the last word, lists the second-figure syllogisms; the final word of that line and the final line list the third-figure syllogisms. The vowels represent the categorical statement forms that make up the syllogism. Thus, *Barbara* consists of three universal affirmatives; *Celarent*, a universal negative and a universal affirmative, with a universal negative conclusion; and so on. The initial consonant specifies the first-figure syllogism to which the syllogism in question reduces. Thus, *Festino* reduces to *Ferio*; *Datisi* reduces to *Darii*. An 's' indicates simple conversion; a 'p,' conversion *per accidens*; an 'm,' a transposition of premises; a 'c,' a *reductio ad absurdum* (that is, by contradiction).

The emergence of such verses might be taken as indicating the importance of Aristotle's deductive method in the theory of syllogisms from a thirteenth-century point of view. But it is more likely just the reverse. Logic students memorized the verse; that removed the need to become skilled at the deductive technique. As Lukasiewicz has shown, Aristotle's deductive method allows one to establish the validity of a wide variety of complex arguments that go far beyond Aristotle's simple forms. Reducing the method to a memorized verse, however, removed it as a living logical technique. It also removed the need for the extensive discussions of metaprinciples governing each figure that occupied most Old Logic texts.

Peter of Spain's *Summulae Logicales*, also known as the *Tractatus*, is perhaps the most influential logic textbook in history.¹⁷ It became the standard textbook in the universities for at least four centuries. It makes or at least marks the theoretical innovations of the New Logicians—a group, often known as *terminists*, that includes William of Sherwood (1190–1249) and Lambert of Auxerre.

Peter's *Tractatus* reviews the rules governing syllogisms in each figure, but, beforehand, does something quite new by proposing rules that apply to syllogisms of any figure. His rules:

1. No syllogism can be made of propositions that are entirely particular, indefinite, or singular.

NON EST LAC VIA MEL VAS ERB ARC/ REN ERM RAC OBD/ EVA NEC AVT ESA DVC NAC, do not follow an obvious code and convey much less information than the more complex expressions that succeeded them.

¹⁶Peter of Spain 1972, 52. This appears unaltered in Buridan 2001, 320, except that he adds punctuation after 'Frisesomorum' and 'Barocho' to signal the shift from one figure to another. Kretzmann 1966, 66n, guesses that William of Sherwood may have invented this verse, though some of the terms appear to have been in use earlier.

¹⁷Historians disagree about who Peter of Spain was; a leading theory is that he became Pope John XXI. For issues surrounding the text, see de Rijk 1968, 1972, and 1982. Peter probably wrote the *Tractatus* around 1240; see de Rijk 1972, lvii.

2. No syllogism in any figure can be made of propositions that are entirely negative.
3. If one of the premises is particular, the conclusion must be particular; but not conversely.¹⁸
4. If one if the premises is negative, the conclusion is negative; and conversely.
5. The middle term must never be placed in the conclusion.

These rules plainly do not suffice to determine the validity or invalidity of any argument of generally syllogistic form. They narrow the possible syllogistic moods to AAA, EAE, AEE, AII, IAI, AOO, OAO, EIO, IEO, and the subaltern moods AAI, AEO, and EAO. But they say nothing about figure; they do not distinguish major terms and premises from minor terms and premises. 1AEE, for example, satisfies all these rules but fails to be valid. (“Every *M* is *P*; No *S* is *M*; ∴ No *S* is *P*.”) The rules are nevertheless important, for they mark the beginning of a quest for a complete set of rules capable of serving as a decision procedure for syllogisms.

Peter’s *Tractatus* outlines the doctrine of distribution. His definition of distribution is somewhat obscure: “Distribution is the multiplication of a common term effected by a universal sign” (“*Distributio est multiplicatio termini communis per signum universale facta*” (209)). Lambert of Auxerre reverses the metaphor: “Distribution is the division of one thing into divided [parts]” (139). Both, however, see distribution as something a determiner such as ‘all’ does to a common noun. They distinguish collective from distributive readings, and observe that universal affirmative determiners (‘all,’ ‘every,’ ‘each,’ etc.) distribute the subject term but not the predicate term, while universal negative determiners distribute both terms. Particular determiners do not distribute their subject or predicate terms. Negated terms reverse in distribution, however, so the predicate of a particular negative proposition is distributed. The intuitive significance of distribution is that distributed terms say something about everything in their extensions. To say that every man is an animal is to say something about each and every man. To say that no animal is a stone is to say something about every animal and also about every stone.

Both Peter and Lambert recognize that distribution plays a role in syllogistic reasoning. There can be no syllogism without a universal premise, they note. But not until the fourteenth century does anyone see how to use distribution to extend Peter’s rules to a decision procedure.

2.3 The Mature Logic of Terms

Walter Burley (also Burleigh; 1275–1344) develops a theory of consequences relevant mostly to the history of the logical connectives.¹⁹ But he links his theory to distribution in a way important to the development of the doctrine, for he sees how distribution matters to syllogistic reasoning. Burley thinks first about conversion. Universal negatives convert simply; universal affirmatives do not. Similarly, particular affirmatives convert

¹⁸Oddly, Peter lists this rule twice.

¹⁹For more on Burley and fourteenth-century logical developments, see Geach 1962, Ockham 1974, 1980, King 1985, Normore 1999, Spade 2002, and Dutilh-Novaes 2008.

simply; particular negatives do not. Universal negatives and particular affirmatives are contradictories, but have something in common: their subject and predicate terms agree in distribution. In 'No *S* is *P*,' both *S* and *P* are distributed. In 'Some *S* is *P*,' both *S* and *P* are undistributed. Universal affirmatives convert *per accidens*; from 'Every *S* is *P*' we can infer 'Some *P* is *S*.' This suggests to Burley a pattern. When terms remain unchanged in distribution, or go from distributed to undistributed, the inference works; when they go from undistributed to distributed, it fails. There is good reason for this: one cannot conclude something about the entire extension of a term on the basis of something pertaining to only part of its extension.

He captures this in a general rule.

Whenever a consequent follows from an antecedent, the distribution of the antecedent follows from the distribution of the consequent.

To put this another way, any term distributed in the conclusion must be distributed in the premises. If we add this rule to those of Peter of Spain (omitting his last as implied by an argument's being of proper syllogistic form), we get

1. No syllogism can be made of propositions that are entirely particular, indefinite, or singular.
2. No syllogism in any figure can be made of propositions that are entirely negative.
3. If one of the premises is particular, the conclusion must be particular.
4. If one if the premises is negative, the conclusion is negative; and conversely.
5. Any term distributed in the conclusion must be distributed in the premises.

This is still not a complete set; it allows for the fallacy of the undistributed middle. To complete the rules, one needs in addition to require that the middle term is distributed at least once. This makes good sense, given the thought that distributed terms say something of everything falling under them, while undistributed terms do not. If neither premise says something about everything that falls under the middle term, the argument fails to relate the parts of the middle term's extension relevant to the premises.

John Buridan (1300?–1358?) adds precisely that requirement. In the *Summulae de Dialectica* he merely reviews Peter of Spain's rules, explaining why each should be true. In the *Treatise on Consequence* (Buridan 1976, King 1985), however, he brings the doctrine of distribution to bear on syllogisms, specifying the following rules:

1. No syllogism can be made of propositions that are entirely particular, indefinite, or singular.
2. No syllogism in any figure can be made of propositions that are entirely negative.
3. If one of the premises is particular, the conclusion must be particular.
4. If one if the premises is negative, the conclusion is negative; and conversely.
5. Any term distributed in the conclusion must be distributed in the premises.

6. The middle term must be distributed at least once.

This is a complete set of rules; it approves all and only syllogisms, providing a simple decision procedure.

Buridan does not rest content with devising a decision procedure for syllogistic reasoning. He develops a general theory of infinite terms—that is, terms such as “non-animal,” formed by negating other terms—and extends his theory to syllogistic-like reasoning involving such terms. As we have seen, practitioners of the Old Logic were in a position to do just that, but failed to accomplish it. Buridan does, writing an entire chapter (5.9, “About Syllogisms with Infinite Terms”) on the subject.

Fourteenth-century logicians made other significant advances in our understanding of quantification. They began to recognize inferential properties of determiners that were to become central to the contemporary theory of generalized quantifiers. Burley, for example, states rules for consequences that correspond to characteristic properties of certain kinds of determiners. For example:

A consequence from a distributed superior to its inferior taken with distribution and without distribution holds good, but a consequence taken from an inferior to its superior with distribution does not hold good. (300)

His example: ‘Every animal is running’ implies ‘Every man is running,’ but not conversely. Burley is here noticing that universal affirmative determiners are *antipersistent*: If every S is P , then every T is P , for any T whose extension is a subset of the extension of S .

Burley and Buridan link this to distribution. If something is said of everything that falls under a term, then it is said of each object individually—that, in fact, is Buridan’s definition of distribution²⁰—and of everything that falls under any subset of the term’s extension. They recognized the connection to inference: distributed terms are, in modern parlance, *monotonic decreasing*. Undistributed terms are monotonic increasing: what is said of a part of a term’s extension is also said of a part of any superset. If we use the convenient notation using arrows to indicate monotonic decreasing or increasing effects on terms—the first arrow indicating the subject term, and the second the predicate—then we can summarize the properties of determiners and correlatively of categorical propositions very simply:

Universal affirmative: ‘every,’ ‘all,’ ‘each’— $\downarrow\text{mon}\uparrow$
Universal negative: ‘no’— $\downarrow\text{mon}\downarrow$
Particular affirmative: ‘some’— $\uparrow\text{mon}\uparrow$
Particular negative: ‘some... not’— $\uparrow\text{mon}\downarrow$

This is enough information to deduce all the inferential properties of categorical propositions, provided that terms are nonempty. Otherwise, the distribution of the universal subject terms, which makes them antipersistent (in this notation, $\downarrow\text{mon}$), suggests that they should hold when their subject terms are empty. Admittedly, a null “part” of an

²⁰“Distributive supposition is that in accordance with which from a common term any of its supposita can be inferred separately, or all of them at once conjunctively, in terms of a conjunctive proposition” (264). Buridan thus expresses the idea that universal quantification is a generalized conjunction.

extension is an odd sort of part. But taking this understanding of determiners to its conclusion leads to modern logic's treatment of universal affirmatives and negatives as vacuously true when nothing falls under their subject terms. This breaks the square of opposition, of course, for then such propositions are not contraries; they can be true at the same time.

Burley and Buridan raise a number of issues of fundamental importance for an adequate theory of quantification. They notice, for example, the plethora of determiners in natural language. Among universal affirmatives, Buridan lists 'every,' 'whichever,' 'whoever,' 'whosoever,' 'both,' 'however much,' 'however many,' 'however many times,' 'whatever... like,' 'whenever,' 'wherever,' 'always,' 'perpetually,' 'eternally,' 'howsoever much,' 'anyhow,' 'anywhere,' "etc." (265), and proceeds to discuss the distinctions between them. This may seem to be a minor point. But Buridan begins the process of thinking not just about certain words ('every,' 'no,' 'some') but about a general category of expressions that he recognizes as playing the same logical and grammatical role.²¹

Burley and Buridan also spend considerable time discussing relational predicates. Aristotle was aware of such predicates; he mentions oblique contexts and gives as an example "there is a genus of that of which there is a science, and if there is a science of the good, we conclude that there is a genus of the good" (I, 36). But no one before the fourteenth century paid such inferences much attention. Armed with a theory of supposition, Burley and Buridan address such inferences in detail. For example, Buridan discusses the inferential properties of these in the *Summulae de Dialectica*:

If someone is father of a daughter, then someone is daughter of a father.

(179)

One seeing every donkey is an animal. (273)

Every man's donkey is running. (274, 366)

Every man is seen by some donkey. (274)

Any animal of a king is a horse. (274)

No man sees every donkey. (277)

Every man sees every donkey. (277)

A horse is greater than a man. (277)

A horse is greater than every man. (277)

A horse is greater than the smallest man. (277)

Socrates loves himself. (283)

Socrates acquires something for himself. (283)

Every man likes himself. (286)

Every man sees his own horse. (287)

Socrates rides his own horse. (287)

Socrates is the same height as Plato. (288)

When Socrates arrives, then Plato greets him. (288)

²¹One earlier (twelfth-century) text that pays similar attention to the variety of determiners is the *Ars Emmerana*, which lists, among universal determiners, 'every,' 'whichever,' 'whoever,' 'whosoever,' 'both,' 'always,' 'anytime' [literally, 'every day'], 'everywhere,' 'no,' 'nobody,' 'nothing,' 'never,' 'nowhere,' and 'neither.' Among particular determiners it lists 'some,' 'somebody,' 'sometimes,' 'somewhere,' 'one,' 'another,' and 'someday.' See 154, f. 46rb.

Socrates sees Plato. (298)

This long list of examples may give some sense of how extensive Buridan's discussion of relations is. He also gives many examples of oblique inferences, including:

Every man's donkey is running; every king is a man; therefore, every king's donkey is running. (366)

Every man sees every man; a king is a man; therefore a king sees a king. (367)

No man's donkey is running; a man is an animal; therefore, some animal's donkey is not running. (369)

Any man's donkey is running and any man's horse is running; therefore, [he] whose horse is running [is such that] his donkey is running. (369–70)

Of whatever quality the thing that Socrates bought was, such was the thing he ate; it was a raw thing that Socrates bought; therefore, it is a raw thing that Socrates ate. (370; see 877)

Buridan thinks seriously about issues we would now consider matters of scope in the context of his theory of supposition:

A white man is going to dispute. (293)

Every white man will be good. (301)

A horse was white. (301)

Many examples of scope ambiguities involve intensional contexts. Buridan, foreshadowing Quine (1956, 1960) and Montague (1973, 1974), notes that 'I think of a rose' may be true even if there are no roses.

I recognize a triangle. (279)

I owe you a horse. (279)

I know the one approaching. (279, 294)

The one approaching, I know. (279, 294)

Socrates sees a man. (282, 296)

How Socrates looks, Plato wants to look. (288)

The First Principle Averroes did not believe to be triune. (295)

The First Principle Averroes did believe to be God. (295)

The Triune Averroes believed to be God. (295)

I can see every star. (296)

I think of a rose. (299)

A golden mountain can be as large as Mount Ventoux. (299)

The one creating is of necessity God. (300)

A man able to neigh runs. (302)

If something is sensible, something is sensitive. (182)

If something is knowable, something is cognitive. (182)

A sense is a sense of what is sensed. (179)

What is sensed is sensed by a sense. (179)

Burley and Buridan make another remarkable advance over earlier discussions. They recognize that quantified expressions not only relate terms but introduce “discourse referents,” enabling further anaphora. They are not the first to realize this—there are a few examples in the *Tractatus Anagnini* two centuries earlier—but the attention they pay anaphoric issues is unprecedented.²² They moreover give examples of expressions that are *not* well-formed to show what role certain expressions can and cannot play. Burley, of course, is the source of the famous “donkey” sentence,

Every farmer who owns a donkey beats it.

But this is only one of many. Consider these examples from Buridan:

- An animal is a man and he is a donkey. (283)
- A man runs and he disputed. (283)
- A man is a stone and he runs. (284)
- ‘Mirror’ is a noun and it has two syllables. (284)
- * A man runs and every he is white. (284)
- * an animal is running and no man is that. (285)
- A man is running and that man earlier was disputing. (285)
- an animal is running and it is a man. (286)
- A man runs and another thing is white. (289)

The puzzle these present is that the first conjunct appears to be a categorical proposition, something of the form ‘Some *S* is *P*.’ The proposition conjoined to it, however, cannot be interpreted as another categorical proposition, but contains an anaphor making reference back to something in the first conjunct. Trying to construe the second conjunct as a categorical proposition yields either something with the wrong truth conditions—‘an animal is a man and he is a donkey,’ which is false, is not equivalent to ‘an animal is a man and an animal is a donkey,’ which is true—or something ungrammatical.

3 The Textbook Theories of Quantification

The fourteenth century was unquestionably a high point in the history of logic. The sophistication and subtlety of Burley, Buridan, and Ockham would not be equalled for five hundred years. In fact, logic fell into a rapid and steep decline.

Some of the reasons for the decline were intellectual. The humanism of the Renaissance led writers to disparage theology and everything associated with it, including philosophy and, in particular, logic. The rise of vernacular languages and the decline of Latin, manifested in the works of Dante, Petrarch, Boccaccio, and Machiavelli in Italian, Rabelais in French, Cervantes in Spanish, and Chaucer, Mallory, Marlowe, and

²²*Tractatus Anagnini* (240, 69v) mentions several under the heading *secundum relationem*: ‘Every man is an animal who is capable of laughter; therefore, every man is an animal, and he is capable of laughter’; ‘some animal lives that neither lives nor moves; therefore, some animal lives, though it neither lives nor moves’; ‘something is not an animal that is a man; therefore, something is not an animal, and it is a man’; and ‘Only Socrates is an animal who is Socrates; therefore, only Socrates is an animal, and he is Socrates.’

Shakespeare in English, led to a loss of interest in a tradition that had, since the time of Boethius, found expression almost entirely in Latin. The rise of rhetoric pushed logic aside as largely beside the point of persuasion.

The chief reasons, however, were material. The end of the medieval warm period led to a series of cold winters and cool and rainy summers that devastated farming throughout Europe, choked economic growth, and led to repeated famines. The Black Death first struck Europe in 1348; it killed roughly a third of the population of Europe. University communities in Oxford, Paris, and other areas were especially hard hit. More than half the faculty died. Latin became unpopular, in fact, largely because the plague killed most of the educated people who were fluent in it. The effects on logic were devastating. Serious attention would not be paid to it until the seventeenth century, and the problems occupying Burley, Buridan, and Ockham were forgotten until well into the twentieth.

3.1 The Port-Royal Logic

Antoine Arnauld (1612–1694) and Pierre Nicole (1625–1695) wrote *Logic, or, The Art of Thinking* (Arnauld and Nicole 1662, 1861) at the Port-Royal convent outside Paris, perhaps with some contributions by Blaise Pascal (1623–1662). The book, written in French rather than Latin, revived logic after centuries of neglect.²³ The book expresses traditional Aristotelian logic in the language of Descartes's new way of ideas. From a strictly logical point of view, it contributes few innovations. It may be worth remarking that the *Port-Royal Logic* is probably the first significant logical work in centuries not to contain the word 'donkey.' Its examples are not artificial and restricted to standard logical illustrations, but come from actual texts, chiefly, from the Bible and classical literature. It is rich in linguistic insight, even if somewhat sparse from a logical point of view.

One logical innovation, however, is important: the distinction between comprehension, or, as logicians writing in English have generally called it, *intension*, and *extension*.

Now, in these universal ideas there are two things, which it is very important accurately to distinguish: COMPREHENSION and EXTENSION. I call the COMPREHENSION of an idea, those attributes which it involves in itself, and which cannot be taken away from it without destroying it; as the comprehension of the idea triangle includes extension, figure, three lines, three angles, and the equality of these three angles to two rigid angles, I call the EXTENSION of an idea those subjects to which that idea applies, which are also called the inferiors of a general term, which, in relation to them, is called superior, as the idea of triangle in general extends to all the different sorts of triangles. (1861, 49)

It is natural to think of this as the origin of our idea of a term's extension as the set of things of which it is true. Arnauld and Nicole, however, lack the concept of a set;

²³This is not quite fair; in the seventeenth century, Leibniz engaged in highly original logical work, but its distribution was so limited that it had little impact. Leibniz himself was influenced by Joachim Jungius, whose *Logica Hamburgensis* (1638, 1977) combined Aristotelian and Ramist approaches.

they use the plural (“those subjects to which the idea applies”). Arguably, therefore, they make only a small advance over the fourteenth-century logicians talk of *supposita*. Their talk of “inferiors” and of “sorts of triangles” moreover raises the possibility that they have in mind not, or not only, the objects to which the term applies but the subsets of what we would today call the extension. The language of inferiors and superiors stems ultimately from Porphyry—indeed, Peirce ridicules Baines’s position that Arnauld and Nicole invented the distinction between intension and extension, crediting it to Porphyry (1893, 237–238) and finding it in Ockham (241–242)—and has the disadvantage of blending these ideas together, obscuring the distinction between members and subsets. That said, first drawing the distinction between intension and extension, however fuzzily, and providing labels for them constitutes an important achievement.

Arnauld and Nicole avoid speaking of distribution, but the idea clearly motivates an argument that all propositions must be either particular or universal:

But there is another difference of propositions which arises from their subject, which is according as this is universal, particular, or singular. For terms, as we have already said in the First Part, are either singular, or common, or universal. And universal terms may be taken according to their whole extension, by joining them to universal signs, expressed or understood: as, *omnis*, all, for affirmation; *nullus*, none, for negation; all men, no man.

Or according to an indeterminate part of their extension, which is, when there is joined to them *aliquis*, some, as some man, some men; or others, according to the custom of languages. Whence arises a remarkable difference of propositions; for when the subject of a proposition is a common term, which is taken in all its extension, propositions are called universal, whether affirmative, as, ‘Every impious man is a fool,’ or negative, as, ‘No vicious man is happy.’

And when the common term is taken according to an indeterminate part only of its extension, since it is then restricted by the indeterminate word ‘some,’ the proposition is called particular, whether it affirms, as, ‘some cruel men are cowards,’ or whether it denies, as, ‘some poor men are not unhappy.’ (110)

Medieval discussions of distribution do not mark the alternatives so starkly or explicitly, though the terms ‘distributed’ and ‘undistributed’ do sound exhaustive.²⁴ As Arnauld and Nicole see it, a term must be either “taken according to [its] whole extension” or not, and thus the subject term must be distributed, in which case the proposition is universal, or undistributed, in which case the proposition is particular.

²⁴Fourteenth-century logicians may have assumed that all terms, in the context of a proposition, are taken with respect to all their *supposita* or only some. Thirteenth-century logicians such as Peter of Spain, however, made no such assumption. Distribution, for Peter, is the multiplication of a common term. There is no direct path from a failure of multiplication to the “involves a part” idea that implies the inferential properties of particulars. This is to Peter’s credit, for, as we have seen, “involves the whole” suggests monotonic decreasing behavior, and “involves a part” implies monotonic increasing behavior. But some determiners are neither: ‘the,’ ‘exactly one,’ ‘exactly two,’ ‘most,’ etc.

This seems to leave no place for singular propositions. But Arnauld and Nicole see singulars as reducing to universals or particulars. They continue, arguing that singular terms are to be taken as universals:

And if the subject of a proposition is singular, as when I say, 'Louis XIII took Rochette,' it is called singular. But though this singular proposition may be different from the universal, in that its subject is not common, it ought, nevertheless, to be referred to it, rather than to the particular; for this very reason, that it is singular, since it is necessarily taken in all its extension, which constitutes the essence of a universal proposition, and which distinguishes it from the particular. For it matters little, so far as the universality of a proposition is concerned, whether its subject be great or small, provided that, whatever it may be, the whole is taken entire. (110)

More seriously, the *Port-Royal Logic* has no place for definite articles and numerical quantifiers.

Arnauld and Nicole have an extended and sophisticated discussion of indefinites (or bare plurals). Far from reducing them to particulars, as Aristotle and his commentators generally do, or to universals, they point out that neither will do. The existence of white bears and English Quakers is not enough, they observe, to make 'Bears are white' and 'Englishmen are Quakers' true. They distinguish moral from metaphysical universality:

We must distinguish between two kinds of universality, the one, which may be called *metaphysical*, the other *moral*.

We call universality, *metaphysical*, when it is perfect without exception, as, *every man is living*, which admits of no exception.

And universality, *moral*, when it admits of some exception, since in moral things it is sufficient that things are generally such, *ut plurimum*, as, that which St. Paul quotes and approves of: "*Cretenses semper mendaces, malae bestiae, centres pigri*" [Cretans are always liars, evil beasts, great gluttons (Titus 1:12)]. Or, what the same apostle says: "*Omnes quae sua sunt quaerunt, non quae Jesu-Christi*" [Everyone seeks his own [good], not that of Jesus Christ (Philippians 2:21)]; Or, as Horace says: "*Omnibus hoc vitium est cantoribus, inter amicos Ut nunquam inducant animum cantare rogati; Injussi nunquam desistant*" [It is a flaw of every singer, among friends, that, if asked to sing, they will never start; if not asked, will never stop (Satire III)]; Or, the common aphorisms:

That all women love to talk.

That all young people are inconstant.

That all old people praise past times.

It is enough, in all such propositions, that the thing be commonly so, and we ought not to conclude anything strictly from them. (147)

Moral universality differs from universality as usually understood, for it does not license universal instantiation:

For, as these propositions are not so general as to admit of no exceptions, the conclusion may be false, as it could not be inferred of each Cretan in particular, that he was a liar and an evil beast....

Thus the moderation which ought to be observed in these propositions, which are only morally universal, is, on the one hand, to draw particular conclusions only with great judgment, and, on the other, not to contradict them, or reject them as false, although instances may be adduced in which they do not hold.... (147–148)

Arnauld and Nicole are content with “moral universality” for contingent (and politically incorrect) generalizations:

Whence we may very well say, *The French are brave; the Italians suspicious; the Germans heavy; the Orientals voluptuous*; although this may not be true of every individual, because we are satisfied that it is true of the majority. (152)

That last remark suggests that bare plurals are equivalent to noun phrases with the determiner ‘most.’ But that it is not quite right. Arnauld and Nicole point out some sentences that are difficult to understand even by such a relaxed criterion, and have become known in the linguistics literature as Port-Royal sentences:

There are also many propositions which are morally universal in this way only, as when we say, *The French are good soldiers, The Dutch are good sailors, The Flemish are good painters, The Italians are good comedians*; we mean to say that the French who are soldiers, are commonly good soldiers, and so of the rest. (150)

Arnauld and Nicole present the square of opposition uncritically, with no consideration of empty terms (112–113). They decline to present rules for the Aristotelian reduction of syllogisms, “because this is altogether useless, and because the rules which are given for it are, for the most part, true only in Latin” (113)! They do, however, give rules derived from those of Buridan for syllogisms:

1. The middle term cannot be taken twice particularly, but it ought to be taken, once at, least, universally.
2. The terms of the conclusion cannot be taken more universally in the conclusion than they are in the premises.
3. No conclusion can be drawn from two negative propositions.
4. A negative conclusion cannot be proved by two affirmative propositions.
5. The conclusion always follows the weaker part, that is to say, if two propositions be negative, it ought to be negative, and if one of them be particular, it ought to be particular.
6. From two particular propositions nothing follows.

Their only innovation is the rigorous demonstration of corollaries from these rules:

- There must always be in the premises one universal term more than in the conclusion.
- When the conclusion is negative, the greater term must necessarily be taken generally in the major.
- The major (proposition) of an argument whose conclusion is negative, can never be a particular affirmative.
- The lesser term is always in the conclusion as in the premises.
- When the minor is a universal negative, if we wish to obtain a legitimate conclusion, it must always be general.
- The particular is inferred from the general.

Arnauld and Nicole also recognize a fourth figure and set out the valid moods in each figure on the basis of the rules and corollaries. They use the medieval names for the moods, but mention the significance only of the vowels, since they reject the project of reducing them to the first figure.

3.2 The English Textbook Tradition

The eighteenth century—the century of Bach, Handel, Haydn, and Mozart, of Burns, Sterne, Fielding, and Goethe, of Berkeley, Rousseau, Hume, and Kant—yielded almost no significant work in logic. Perhaps the most important work, Isaac Watts’s *Logic, or, The Right Use of Reason* (1725), is a work of metaphysics and epistemology modeled on Locke’s *Essay* more than a work of logic in any traditional sense. Watts follows Arnauld and Nicole in his rather brief treatment of inference, as well as in his discussion of moral universality and Port-Royal sentences (120–21). He discusses categorical propositions and conversion in a page or two, and offers the Port-Royal rules for syllogisms, but with a mistake, stipulating that “if either of the premises be negative, the conclusion must be particular.” Perhaps the most interesting aspect of Watts’s text is his discussion of relative and connective syllogisms, such as

As is the captain so are his soldiers.
The captain is a coward.
Therefore, his soldiers are so too. (233)

Meekness and humility always go together.
Moses was a man of meekness.
Therefore, Moses was a man of humility. (234)

London and Paris are in different latitudes.
The latitude of London is 51.5 degrees.
Therefore, this cannot be the latitude of Paris. (234)

The first of these is interesting not only in being relational but also in being higher-order.

In the nineteenth century, British textbooks revived an interest in logic. Initially, they did little more than review the traditional theory of the syllogism, at a level of detail and sophistication comparable to that of the early twelfth century or the *Port-Royal Logic*. But Richard Whately (1787–1863) wrote *Elements of Logic* (1826), the most extensive discussion of logical subjects since the fourteenth century. The book sets an organizational pattern—language, deductive logic (categorical propositions, conversion, syllogisms, hypotheticals and other connectives), fallacies, inductive reasoning, questions of method—that continues in most contemporary logic textbooks. It moreover contains some significant innovations. Whately, “the restorer of logical study in England,” in the words of DeMorgan (quoted in Valencia 2004, 404), defines a proposition simply as an indicative sentence. In sharp contrast to the *Port-Royal Logic*, he discusses distribution, and in fact uses it to define universal and particular “signs,” i.e., determiners. The subjects of universal sentences are distributed in the sense that they “stand, each, for the whole of its Significates” while particular subjects “stand for a part only of their Significates” (76). This is the doctrine of distribution that Peter Geach famously attacks in *Reference and Generality* (1962), that distributed terms refer to all, and undistributed terms some, of the objects falling under them.²⁵

It is worth noting, however, that this way of putting the doctrine is not inevitable. It is not quite a nineteenth-century invention; Ockham treats universal signs as “mak[ing] the term to which it is added stand for all its significata and not just for some” (1980, 96). The dominant fourteenth-century conception, however, is inferential: a term is distributed in a proposition if and only if that proposition implies all its substitution instances with respect to that term. Thus, in ‘every man is mortal,’ ‘man’ is distributed, for the sentence implies, for each individual man, that he is mortal. The idea is that ‘Every man is mortal’ and ‘Socrates is a man’ imply ‘Socrates is mortal.’ For particular terms, the pattern reverses; from ‘Socrates is a man’ and ‘Socrates is mortal’ we can infer ‘Some man is mortal.’ There is no temptation, on that inferential conception, to think that every term must be universal or particular, or that one is forced to any specific thesis about the semantic values of terms.

Whately discusses bare plurals only briefly. They must, he reasons, be either universal or particular; he sees no other option. But they differ from noun phrases with a universal or particular determiner in that their significance depends on the context and content of the sentence in which they appear. That is an important insight.²⁶ But his illustrations are not very helpful. ‘Birds have wings’ and ‘birds are not quadrupeds’ he takes to be universal; ‘food is necessary to life,’ ‘birds sing,’ and ‘birds are not carnivorous,’ as particular.

Whately uncritically describes the square of opposition, with the standard diagram, but his discussion of conversion links it to distribution. His rule is, “No term is distributed in the Converse, which was not distributed in the Exposita” (82). Since conversion switches terms, this means that categorical propositions in which the subject and predicate terms agree in distribution—that is, particular affirmatives and univer-

²⁵See Geach 1962, and, for a response, King 1985.

²⁶See, for example, Carlson 1977.

sal negatives—convert simply, but universal affirmatives convert only *per accidens*, and particular negatives convert only by contraposition. His rule also explains why universal negatives also convert *per accidens*, and universal affirmatives convert by contraposition.

Whately is perhaps the first logician to think of categorical propositions as concerned with classes:

...in the first Premiss (“X is Y”) it is assumed universally of the *Class* of things (whatever it may be) which “X” denotes, that “Y” may be affirmed of them; and in the other Premiss (“Z is X”) that “Z” (whatever it may stand for) is referred to that Class, as comprehended in it. Now it is evident that whatever is said of the whole of a Class, may be said of anything that is comprehended (or “included,” or “contained,”) in that Class: so that we are thus authorized to say (in the conclusion) that “Z” is “Y”. (86; quoted in Hailperin 2004, 343)

The psychologistic talk of ideas introduced by Arnauld and Nicole thus yielded to talk of classes, which made possible the advances of the nineteenth and twentieth centuries. Augustus DeMorgan and George Boole both read Whately; whether the idea that logic concerns relations of classes was Whately’s insight or something “in the air” in nineteenth-century Britain is hard to say.

Perhaps Whately’s most important contribution, however, is the modern conception of deductive validity. Aristotle, recall, defines validity in terms of the conclusion following necessarily from the premises. Medieval discussions of conversion and other forms of inference employ a notion of truth-preservation, but do not invoke it explicitly to characterize argumentative success. Whately does:

... an argument is an expression in which *from something laid down and granted as true (i.e. the premises) something else (i.e. the Conclusion,) beyond this must be admitted as true, as following necessarily (or resulting) from the other...* (86)

Because logic is concerned with language, moreover, Whately observes that logical validity is a matter of form rather than content, in which we consider “the mere *form of the expression*” rather than “the meaning of the terms” (86).

Whately cites Aristotle for the *dictum de omni et nullo*, which he phrases in terms of distribution: “whatever is predicated of a term distributed, whether affirmatively or negatively, may be predicated in like manner of everything contained under it” (87). He takes this, immediately, as the principle underlying first-figure syllogisms, and only mediately as the principle of all syllogisms. He speaks of the danger of equivocation on the middle term, and uses it as an argument for the rule that the middle must be distributed at least once. By a similar argument, he justifies the rule that no term may be distributed in the conclusion that was not distributed in the premises. He gives two other rules: from negative premises, nothing follows, and if a premise is negative, the conclusion must be negative, and vice versa. Whately is thus the source of the rules for syllogisms as they are sometimes given in contemporary textbooks:

1. The middle term must be distributed at least once.

2. No term may be distributed in the conclusion without being distributed in the premises.
3. Nothing follows from two negative premises.
4. If a premise is negative, the conclusion must be negative, and vice versa.

He observes that the other standard rules, namely, that from two particular premises nothing follows, and if a premise is particular, the conclusion must be particular, follow from these rules. Whately gives illustrations rather than arguments for these observations. But he is right.²⁷ Unlike previous logicians, including Arnauld and Nicole, he does not neglect subaltern moods, but notes that “when we can infer a universal, we are always at liberty to infer a particular” (91). Whately goes through the valid syllogisms, showing that each satisfies the rules and moreover showing, in detail, how to reduce each to first-figure syllogisms.

3.3 Quantifiers in the Predicate

In the middle of the nineteenth century, Augustus de Morgan and Sir William Hamilton argued publicly about one of the most peculiar ideas in the history of logic—the idea that the logical form of categorical propositions includes quantification governing the predicate as well as the subject. Since Aristotle, quantifiers had been conceived as relations between terms. The doctrine of distribution, however, suggests another way of thinking about quantifiers.

The idea of quantification of the predicate occurs in William of Ockham (1287–1347), one of the greatest medieval philosophers, who shares with Buridan the thought that universality is a generalization of conjunction and particularity a generalization of disjunction.²⁸ Suppose that we can enumerate the *S*s as s_1, \dots, s_n and the *P*s as p_1, \dots, p_m . Then ‘Every *S* is *P*’ is equivalent to ‘ s_1 is *P* and ... and s_n is *P*.’ But we can go further. If s_1 is *P*, it must be one of the *P*s; so, $s_1 = p_1$ or ... or $s_1 = p_m$. We can do this for each of the *S*s. So, ‘Every *S* is *P*’ is equivalent to

²⁷The omitted arguments are somewhat interesting, and use all four rules. To show that nothing follows from two particular premises: Suppose that an argument of syllogistic form has two particular premises. Then both subject terms must be undistributed. Since the middle term must be distributed at least once (Rule 1), at least one of the predicate terms must be distributed. At least one of the premises, then, must be negative, so the conclusion must be negative as well (Rule 4). But then the major term is distributed in the conclusion, so it must be distributed in the major premise (Rule 2). Since the subject term of the major is undistributed, the major term must be the predicate of the major premise, making it negative. But then both premises are negative, so nothing follows (Rule 3).

To show that if a premise is particular, the conclusion must be particular, assume that a premise is particular. By the previous result, the other must be universal. So, one subject term is distributed, while the other is undistributed. Suppose that the conclusion is universal, so that its subject term is distributed. Then the minor term must be distributed in the premises (Rule 2). The middle must be distributed at least once (Rule 1), so the premises must contain at least two distributed terms, only one of which may be in the subject position. So, one of the premises must be negative. It follows that the conclusion must also be negative (Rule 4). So, the major, being distributed in the conclusion, must be distributed in the premises (Rule 2). But that means that there must be three distributed terms in the premises, only one of which appears in subject position. But that means both premises must be negative, and nothing follows from two negative premises (Rule 3).

²⁸Ockham may have been inspired by reflecting on a fourteenth-century sophism. Here, for example, is William Heytesbury (1335): “Every man is every man. Proof: This man is this man, and that man is that man, and thus for each one. Therefore every man is every man” (Wilson 1960, 154; my translation).

$(s_1 = p_1 \text{ or } \dots \text{ or } s_1 = p_m)$ and $(s_2 = p_1 \text{ or } \dots \text{ or } s_2 = p_m)$ and ... and
 $(s_n = p_1 \text{ or } \dots \text{ or } s_n = p_m)$.

This, in turn, is equivalent to 'Every S is some P .' That suggests the possibility of quantifying the predicate term explicitly, generating additional basic forms (1980, 101), only some of which Ockham discusses:

Every S is every P
Every S is some P
Some S is every P
Some S is some P
No S is every P
No S is some P
Some S is not every P
Some S is not some P

To understand the meanings of these forms, simply place conjunctions (for 'every') or disjunctions (for 'some') in the above schema. The first thus becomes ' $(s_1 = p_1$ and ... and $s_1 = p_m)$ and $(s_2 = p_1$ and ... and $s_2 = p_m)$ and ... and $(s_n = p_1$ and ... and $s_n = p_m)$,' which is true iff there is exactly one S , and exactly one P , and those things are identical. The second is a universal affirmative; the third asserts that there is exactly one P , and it is S . The fourth is a particular affirmative, 'Some S is P .'

To understand the negatives, treat them as contradictories of corresponding affirmatives. The sixth and eighth are thus equivalent to the familiar universal and particular negatives. The fifth denies 'Some S is every P ,' and the seventh, 'Every S is every P .'

Hamilton's idea of quantification in the predicate is not Ockham's. What distinguishes the four kinds of categorical propositions, Hamilton thinks, is the distribution of subject and predicate terms. Think of a determiner as a *predicate* of a term, indicating its distribution or lack thereof. Then we would need to mark the distribution (or not) of the subject as well as the distribution (or not) of the predicate. Hamilton sees conversion as forcing recognition of quantification in the predicate. He writes,

The second cardinal error of the logicians is the not considering that the predicate has always a quantity in thought, as much as the subject, though this quantity frequently be not explicitly enounced.... But this necessity recurs, the moment that, by conversion, the predicate becomes the subject of the proposition; and to omit its formal statement is to degrade Logic from the science of the necessities of thought, to an idle subsidiary of the ambiguities of speech. (1860, quoted in Bochenski 1961, 263–64)

This is puzzling, since, traditionally, the quality of the proposition as affirmative or negative, not the quantity (as universal or particular) specified by the determiner, fixes the distribution of the predicate term. In any case, Hamilton thinks of the basic statement forms as

All S is all P
All S is some P
Some S is all P

Some S is some P
 Any S is not any P
 Any S is not some P
 Some S is not any P
 Some S is not some P

These make little grammatical or logical sense. (Why the singular copula ‘is’ with the plural subject ‘all S ’? Why the switch from ‘all’ to ‘any’?) Evidently the usual understanding of the universal affirmative is ‘All S is some P ’; of the particular affirmative, ‘Some S is some P .’ But what do the other forms mean? What are their negations? Which contradict which? The generally distribution-driven motivation suggests that there are only four statement forms to be recognized, for each of the two terms may be distributed or undistributed. What is left for the other forms to do? Despite that rather obvious puzzle, logicians of the stature of Boole (1848) and De Morgan (1860) adopted Hamilton’s idea.

Biologist George Bentham had developed this approach two decades before Hamilton, and somewhat more clearly.²⁹ He evidently thought of terms collectively, and thought of categorical propositions as relating all or part of one collective to all or part of another. The only relations he recognized, however, were identity and diversity. So, to simplify his notation somewhat, letting p represent part, the forms become

$S = P$
 $S = pP$
 $pS = P$
 $pS = pP$
 $S \neq P$
 $S \neq pP$
 $pS \neq P$
 $pS \neq pP$

Imagine diagramming these forms as relations between sets, with each set represented by a circle. We might interpret the first as saying that the circle representing the S s just is the circle representing the P s—that is, that S and P have the same extension. We might interpret the second as holding that the S s are a subset of the P s, and the third as holding the converse. The fourth we might interpret as holding that the extensions of S and P overlap. The next four would be the negations of these. Bentham’s scheme thus has the advantage of making it clear what these forms mean. Unfortunately, it also makes clear its pointlessness. All these relations are easily expressible in traditional terms:

Every S is P and every P is S
 Every S is P
 Every P is S
 Some S is P
 Some S is not P or some P is not S
 Some S is not P

²⁹George Bentham, *Outline of a New System of Logic*, 1827.

Some P is not S
 No S is P

Bentham's notation might nevertheless be justified if it facilitated a simple method for determining validity. Neither Bentham nor Hamilton developed any such method. Still, given a logic of identity, and the principle that a part of a part is a part ($ppX = pX$), it is not hard to do. This, for example, is *Barbara*:

Every M is P	$M = pP$
Every S is M	$S = pM$
\therefore Every S is P	$S = pP$

The reasoning: $S = pM = ppP = pP$. *Celarent*:

No M is P	$pM \neq pP$
Every S is M	$S = pM$
\therefore No S is P	$pS \neq pP$

This too admits of easy proof: $pS = ppM = pM \neq pP$. *Darii*:

Every M is P	$M = pP$
Some S is M	$pS = pM$
\therefore Some S is P	$pS = pP$

The proof: $pS = pM = ppP = pP$. Finally, *Ferio*:

No M is P	$pM \neq pP$
Some S is M	$pS = pM$
\therefore Some S is not P	$S \neq pP$

This proof is only slightly trickier: say $S = pP$. Then $pS = ppP = pP$ and $pS = pM$, so $pM = pP$, contrary to the first premise. So, $S \neq pP$.

Immediate inferences, too, are straightforward. The notation makes it obvious that A and O forms, like I and E forms, are contradictories. A and E forms are contraries, since A forms imply I forms; if $S = pP$, $pS = ppP = pP$. I and O forms are subcontraries: one or the other must both be true. Say the O form is false, so that $S = pP$. Then $pS = ppP = pP$, and the I form is true. Simple conversion rules follow immediately from the symmetry of identity.

Though the position of Bentham and Hamilton seems ill-motivated and even confused from a semantic point of view, therefore, it does yield a strikingly simple method for demonstrating the validity of syllogisms. It requires some elaboration, however, to become a way of showing invalidity—as it stands, the method is unsound—or even establishing validity within the formal language. One supplement is a principle of extensionality: $S = pP \wedge P = pS \Rightarrow S = P$. Another is to elaborate the part function into a family of functions, stipulating that if a premise would employ an *identity* statement with a term pX , or the conclusion would employ a *diversity* statement with a term pX , that has already appeared in an identity statement in the premises, write instead $p'X$, to indicate a *different* part of X . To see why this matters, consider a simple case of undistributed middle, 2AAI. In the unelaborated notation, it becomes

$P = pM, S = pM; \therefore pS = pP$. This is valid; in fact, the premises imply $S = P$. As elaborated, however, it becomes $P = pM, S = p'M; \therefore pS = pP$, which fails. Similarly, consider an invalid argument cited above: ‘Every M is P ; No S is M ; \therefore No S is P .’ The premises become $M = pP$ and $pS \neq pM$; we must represent the conclusion as $pS \neq p'P$, lest we get a form easily shown to be valid.

Though the Bentham/Hamilton doctrine of quantification in the predicate today strikes us as strange, it did contribute two ideas that influenced subsequent algebraic logicians. The first is that a proposition is an equation; the second, that a symbol for a part of a class—something Boole would call an *elective* symbol—could represent existence. Few logicians today would find either idea meritorious. But they played an important role in the development of nineteenth-century logic.

4 The Rise of Modern Logic

Most turning points in the history of ideas are less sharply defined than they first appear. Those who make significant advances frequently worked in a setting in which a variety of other thinkers were working on similar problems and might easily have made the discovery first. Not so with modern logic, which has a well-defined starting point. Modern logic began in 1847, when George Boole (1815–1864) published his short book, *The Mathematical Analysis of Logic, being an Essay towards a Calculus of Deductive Reasoning*. He followed it with *An Investigation of the Laws of Thought, on Which are Founded the Mathematical Theories of Logic and Probabilities* in 1854. Boole’s work displays several advances of vital significance in our understanding of logic and, specifically, of quantification. Boole himself lists them in his 1848 paper, “The Calculus of Logic”:

1. *That the business of Logic is with the relations of classes, and with the modes in which the mind contemplates those relations.* Seeing logic as resting on a theory of classes not only makes mathematical logic possible but also gives logicians tools to create formal theories of syntax, semantics, and pragmatics. Medieval logicians spoke of a term’s *supposita*; the Port-Royal Logic, of a term’s extension. But both of these were conceived in the plural, even if the grammatical singular ‘extension’ suggested otherwise. Thinking of a term’s *supposita* or extension as an entity, a class, governed by mathematical laws was something altogether new when Whately introduced it about two decades before Boole. Boole was the first to use the insight as the basis of a logical theory. That brings us to Boole’s second point:
2. *That antecedently to our recognition of the existence of propositions, there are laws to which the conception of a class is subject,—laws which are dependent upon the constitution of the intellect, and which determine the character and form of the reasoning process.* Georg Cantor is rightly viewed as the founder of set theory. But the *idea* of a theory of classes is already in Boole, and he goes some way toward developing it.
3. *That those laws are capable of mathematical expression, and that they thus constitute the basis of an interpretable calculus.* Boole not only conceives of a

calculus of classes—something that might, arguably, be attributed to Leibniz almost two centuries earlier—but constructs one that forms a part of Cantor’s set theory and continues to be used today.

4. *That those laws are, furthermore, such, that all equations which are formed in subjection to them, even though expressed under functional signs, admit of perfect solution, so that every problem in logic can be solved by reference to a general theorem.* Boole achieves a higher degree of abstraction in his calculus than contemporary logical theories do, for he brings propositional logic, quantification theory, and abstract algebra together in a single theory. Thus:
5. *That the forms under which propositions are actually exhibited, in accordance with the principles of this calculus, are analogous with those of a philosophical language.* Boolean algebra continues to be an important field of investigation, underlying not only propositional logic but the theory of computation. The idea of a proposition as a class—of “cases,” as Boole puts it, or possible worlds, or indices—continues to be influential in formal semantics.
6. *That although the symbols of the calculus do not depend for their interpretation upon the idea of quantity, they nevertheless, in their particular application to syllogism, conduct us to the quantitative conditions of inference.* Boole’s calculus is not only a propositional logic but a theory of quantification. The classes it deals with may be conceived as classes of cases in which a proposition is true or as classes of objects of which a term is true. The thought that the problems of the relations between propositions and the relations between terms are essentially the same problem is a profound insight that underlies the unification of the two in modern predicate logic, though it also lies deeper, and remains independent of the particular fashion in which that theory combines them.

4.1 Algebraic Approaches to Quantification

Since the systems of Boole 1847 and 1854 have been subjects of extensive discussion—see, for example, Hailperin 2004 and Valencia 2004—I shall focus here on Boole 1848, an intermediate approach that has received less attention. Boole’s calculus starts with a universe of discourse, represented by the numeral 1. We might think of that as a class of objects, or of cases, or of possible worlds; the theory is independent of any particular conception of what the universe of discourse contains. Everything else is in effect a subclass, “formed by limitation” from that universe.

Concatenating two class symbols denotes the intersection of the classes: xy is the class of things in both x and y .³⁰ Thus, $x1 = x$. $1 - x$ is the class of all things in the universe of discourse that are not in x . So, $x(1 - y)$ is the class of things in x but not y ; $(1 - x)(1 - y)$ is the class of thing in neither x nor y . Boole uses addition to represent disjunction or union.³¹

³⁰Boole equivocates on what variables such as x and y denote. Most of the time, he thinks of them as class symbols, but he also refers to them as operations.

³¹Boole introduces this operation with the equation $x(y + z) = xy + xz$; he never defines the operation itself. He seems to think of disjunction as exclusive, though most of what he says applies as well if we understand

Boole endorses these algebraic laws governing his operations:

$$x(y + z) = xy + xz$$

$$xy = yx$$

$$x^n = x$$

The last is the most striking, and differentiates his calculus from usual algebraic theories.³²

Boole turns to the analysis of categorical statement forms. The contemporary reader might guess at what is coming next. If we were to define $0 = 1 - 1$:

- All Ys are Xs: $y = yx$
- No Ys are Xs: $yx = 0$
- Some Ys are Xs: $yx \neq 0$
- Some Ys are not Xs: $y \neq yx$

But that is *not* what Boole does.³³ Instead, influenced by the “quantification in the predicate” idea of Bentham and Hamilton, he writes,

The expression All Ys represents the class Y and will therefore be expressed by y , the copula are by the sign $=$, the indefinite term, Xs, is equivalent to Some Xs. It is a convention of language, that the word Some is expressed in the subject, but not in the predicate of a proposition. The term Some Xs will be expressed by $v x$, in which v is an elective symbol appropriate to a class V, some members of which are Xs, but which is in other respects arbitrary. Thus the proposition A will be expressed by the equation $y = v x$.

Note the clear implication: universal affirmatives have existential import. What Boole actually offers, therefore, is this:

- All Ys are Xs: $y = v x$
- No Ys are Xs: $y = v(1 - x)$
- Some Ys are Xs: $v y = v' x$
- Some Ys are not Xs: $v y = v'(1 - x)$

disjunction as inclusive. His equation holds for both interpretations.

³²Boole uses, but never states explicitly, laws of associativity, even though they had already been stated and named by Hamilton, among others.

³³He does represent universal affirmatives as $y = yx$ in Boole 1847, but immediately moves to an alternative representation $y(1 - x) = 0$. In 1847 he represents particular affirmatives as $xy = v$, where v is an elective symbol representing, evidently, some nonempty class. In 1848, however, he represents universal affirmatives as $y = v x$. Since he denies that universals have existential import, denying the traditional thesis of subalternation, he evidently intends to allow for the possibility that v is empty. But that makes nonsense of his representing particular affirmatives as $v y = v' x$.

where v' is another arbitrary class. This should look familiar; $v, v',$ etc., are functioning much as $p, p',$ etc., function in the Bentham-inspired system described above. He sees that he has the tools to handle everything in the expanded syllogistic language with infinite terms, and gives these equivalences:

- All not- Y s are X s: $1 - y = vx$
- All not- Y s are not- X s: $1 - y = v(1 - x)$
- Some not- Y s are X s: $v(1 - y) = v'x$
- Some not- Y s are not- X s: $v(1 - y) = v'(1 - x)$

Boole demonstrates the validity of contraposition, arguing for the equivalence of $y = vx$ and $1 - x = v(1 - y)$. His method of development for elective equations, as he calls it, is highly general but quite complex, amounting to a way of expressing equations in what we now call disjunctive normal form. The method consists in expressing equations as sums of products of coefficients and terms. For a single variable x , the equation has the form

$$\phi(x) = \phi(1)x + \phi(0)(1 - x)$$

For two variables x and y , the equation has the form

$$\phi(xy) = \phi(11)xy + \phi(10)x(1 - y) + \phi(01)(1 - x)y + \phi(00)(1 - x)(1 - y)$$

For three, it has the form

$$\phi(xyz) = \phi(111)xyz + \phi(110)xy(1 - z) + \phi(101)x(1 - y)z + \phi(100)x(1 - y)(1 - z) + \phi(011)(1 - x)yz + \phi(010)(1 - x)y(1 - z) + \phi(001)xy(1 - z) + \phi(000)(1 - x)(1 - y)(1 - z).$$

The general solution to an equation in three variables is

$$z = \frac{\phi(110)}{\phi(110) - \phi(111)}xy + \frac{\phi(100)}{\phi(100) - \phi(101)}x(1 - y) + \frac{\phi(010)}{\phi(010) - \phi(011)}(1 - x)y + \frac{\phi(000)}{\phi(000) - \phi(001)}(1 - x)(1 - y).$$

Notice that the key, in every case, is to substitute 0 and 1 in for the variables to discover the appropriate coefficients.

We can use this last solution to demonstrate the validity of contraposition. Assume $y = vx$. Let $z = 1 - x$; then $y = v(1 - z)$, so $y - v(1 - z) = 0$. Think of this as $\phi(vyz) = 0$. We can compute the value of this function for each combination of 1s and 0s to derive the equation

$$1 - x = v(1 - y) + \frac{1}{0}(1 - v)y + \frac{0}{0}(1 - v)(1 - y)$$

The second and third terms appear difficult to interpret. Say, however, that $\frac{1}{0}(1 - v)y = t$; then $(1 - v)y = 0$. So, the second term simply drops out. The third term cannot be handled so easily; Boole replaces $\frac{0}{0}$ with an arbitrary elective symbol w . The equation thus becomes

$$1 - x = v(1 - y) + w(1 - v)(1 - y) = (v + w(1 - v))(1 - y)$$

Replacing $(v + w(1 - v))$ with another elective symbol u , we obtain

$$1 - x = u(1 - y)$$

which represents ‘All not- X s are not- Y s.’

Strangely, perhaps, syllogistic inferences are less complex than immediate inferences in Boole’s system. Consider *Barbara*:

$$\begin{array}{ll} \text{Every } M \text{ is } P & m = vp \\ \text{Every } S \text{ is } M & s = um \\ \therefore \text{Every } S \text{ is } P & s = wp \end{array}$$

This is very simple; just let $w = uv$. *Darii* is similarly straightforward:

$$\begin{array}{ll} \text{Every } M \text{ is } P & m = vp \\ \text{Some } S \text{ is } M & u's = um \\ \therefore \text{Some } S \text{ is } P & u''s = wp \end{array}$$

Let $u'' = u'$ and $w = uv$. Consider *Celarent*:

$$\begin{array}{ll} \text{No } M \text{ is } P & m = v(1 - p) \\ \text{Every } S \text{ is } M & s = um \\ \therefore \text{No } S \text{ is } P & s = w(1 - p) \end{array}$$

Simply let $w = uv$. Finally, consider *Ferio*:

$$\begin{array}{ll} \text{No } M \text{ is } P & m = v(1 - p) \\ \text{Some } S \text{ is } M & us = u'm \\ \therefore \text{Some } S \text{ is not } P & ws = w'(1 - p) \end{array}$$

Let $w = u$, $w' = u'v$.

Boole’s 1848 technique thus yields a simple way to demonstrate the validity of syllogisms. It evidently assigns universal affirmatives existential import, validating subalternation:

$$\begin{array}{ll} \text{Every } S \text{ is } P & s = vp \\ \therefore \text{Some } S \text{ is } P & us = u'p \end{array}$$

If $s = vp$, $us = uvp$, and, letting $u' = uv$, $us = u'p$. Universal affirmatives and negatives end up as contraries, for, if $s = 0$, $s = vp$ and $s = v(1 - p)$ are both false.

Boole’s systems of 1847, 1848, and 1854 differ in details, but share many of the same strengths and weaknesses. They unite logic and algebra, addressing the problems of the former using the techniques of the latter. They are highly abstract, applying to both predicate and propositional logic. They are in some ways quite elegant. Their handling of elective symbols, however, is far from rigorous. They leave certain key concepts, including addition (disjunction) and subtraction (negation), undefined, while leaving many of the rules governing them implicit. Most seriously, perhaps, they are strictly monadic, finding no place for relations.

4.2 Peirce's Quantification Theory

American philosopher Charles Sanders Peirce (1839–1914) developed a comprehensive theory of quantification that had far less impact than it might have had, due to the relative inaccessibility of much of Peirce's writings. Even within the field of logic, he developed not one logical system but many, making it difficult to assess his overall contributions. One of his systems, however, is essentially modern quantification theory, which he presents in a form almost identical to that used by most twentieth-century logicians. This is no accident; Peirce influenced Schröder, Löwenheim, and Skolem, who took over many aspects of his notation. Indeed, the word 'quantifier' is Peirce's. (See Hilpinen 2004, 615–616.)

Perhaps the best way to begin is with Peirce's concept of a predicate. A proposition, he says, is "a statement that must be true or false" (1893, 208). A predicate is a residue of a proposition: "let some parts be struck out so that the remnant is not a proposition, but is such that it becomes a proposition when each blank is filled by a proper name" (203). So far, this is traditional, though expressed in language very closely to Russell's talk of propositional functions. Peirce's examples would have shocked Aristotle or Arnauld:

Thus, take the proposition "Every man reveres some woman." This contains the following predicates, among others:

"...reveres some woman."

"...is either not a man or reveres some woman."

"Any previously selected man reveres..."

"Any previously selected man is..."

Predication is "the joining of a predicate to a subject of a proposition so as to increase the logical breadth without diminishing the logical depth" (209). He intends this to be vague, admitting different interpretations. But Peirce is concerned to argue that predicate logic encompasses all of logic, subsuming propositional logic under the general heading of quantification theory. Predication is involved in every proposition, even 'It is raining,' which, he argues, has an indexical subject, and in conditionals, "in the same sense that some recognized range of experience or thought is referred to" (209).

Peirce introduces quantification in terms of the traditional syllogistic conception of quantity. "The quantity of a proposition is that respect in which a universal proposition is regarded as asserting more than the corresponding particular proposition: the recognized quantities are Universal, Particular, Singular and—opposed to these as 'definite'—Indefinite" (214). He mentions Hamilton's idea of quantification of the predicate, but dismisses it "as an instructive example of how not to do it" (1893, 323).³⁴ He also dismisses DeMorgan's system of eight quantified propositions. If a proposition's quantity is that by virtue of which a universal proposition asserts more than a particular proposition, the real question is what that is.

³⁴This is only the beginning of his invective: "The reckless Hamilton flew like a dor-bug into the brilliant light of DeMorgan's mind in a way which compelled the greatest formal logician who ever lived to examine and report upon the system" (1893, 324). He calls Hamilton's followers "perfect know-nothings" (324–325), and refers to Hamilton himself as "utterly incapable of doing the simplest logical thinking," "an exceptionally weak reasoner" (326); his doctrines are "of no merit," "glaringly faulty" (345), "all wrong," "ridiculous" (326).

Peirce begins by concurring with Leibniz that a universal proposition neither asserts nor implies the existence of its subject. In fact, his initial example of a universal proposition is ‘Any phoenix rises from its ashes’ (216). He gives two arguments. The first is the *dictum de omni*: what is asserted universally of a subject is “predicable of whatever that subject may be predicable’ (217). That makes no commitment to the existence of anything. The second is that formal logic should have as wide application as possible; a theory allowing empty terms has broader scope than one that restricts itself to terms with nonempty extensions. The third is the square of opposition. Formal logic, Peirce says, needs to specify a negation for each simple proposition. If ‘All inhabitants of Mars have red hair’ makes no existential commitments, then its negation is simply ‘Some inhabitants of Mars do not have red hair.’ If it means ‘There are inhabitants of Mars and every one of them without exception has red hair,’ then its negation must be ‘Either there is no inhabitant of Mars, or if there be, there is one at least who has not red hair’ (217), which is not a categorical proposition. Peirce sees clearly that, if the negation of a categorical proposition is to be another categorical proposition—something maintained by the traditional square of opposition—then there is a straightforward choice: either universals carry existential import, or particulars do, but not both. Pick one. It is natural to pick particulars, since ‘Some inhabitants of Mars have red hair’ and ‘There are inhabitants of Mars with red hair’ seem obviously equivalent. The consequence is that A and O propositions contradict each other, as do E and I propositions. But, Peirce notes (1893, 283), those are the only relations in the square of opposition that actually hold. A and E propositions can both be true if their subject terms are empty; I and O propositions can both be false in the same circumstance. Universal propositions do not entail the corresponding particular propositions, for the latter have an existential import the former lack.

Peirce cautions that we should not take existence as a predicate, citing Kant’s critique of the ontological argument. He says some intriguing things in light of later discussions, developing the concept of a universe of discourse (1893, 281), now often called a contextual domain: “Every proposition refers to some index: universal propositions to the universe, through an environment common to speaker and auditor, which is an index of what the speaker is talking about” (1893, 218).³⁵ He also speaks in a language friendly to game-theoretic semantics and even to intuitionism: “But the particular proposition asserts that, with sufficient means, in that universe would be found an object to which the subject term would be applicable, and to which further examination would prove that the image called up by the predicate was also applicable” (218).

Peirce renders particular and universal propositions in forms now familiar, linking categorical propositions to propositions with quantifiers and sentential connectives:

Some inhabitants of Mars have red hair \Leftrightarrow Something is, at once, an inhabitant of Mars and is red haired
 All inhabitants of Mars have red hair \Leftrightarrow Everything that exists in the universe is, if an inhabitant of Mars, then also red haired

The quantifier is independent of the proposition’s terms, asserting “the existence of a

³⁵Peirce credits this to DeMorgan 1847, 380.

vague something to which it pronounces ‘inhabitant of Mars’ and ‘red haired’ to be applicable” (218). Thus,

A particular proposition is one which gives a general description of an object and asserts that an object to which that description applies occurs in the universe of discourse.... (221)

Peirce commits himself to what would later be called Quine’s dictum—that to be is to be a value of a variable (Quine 1939)—when he declares that ‘Something is non-existent’ “is an absurdity, and ought not to be considered as a proposition at all” (222). The quantifier asserts existence, so ‘Something is non-existent’ is a contradiction. Peirce seems to think that contradictory propositions are not propositions at all. At other places, however, he recognizes that the universe of discourse of a proposition is defined by “the circumstances of its enunciation” (326), and consists of “some collection of individuals or of possibilities”: “At one time it may be the physical universe, at another it may be the imaginary ‘world’ of some play or novel, at another a range of possibilities” (326–327).

Peirce’s (1885) formalism for quantification looks remarkably modern. He uses the material conditional, and adopts as axioms and rules, which he calls, indiscriminately, *icons*:

$$\begin{aligned}
 & x \rightarrow x \\
 & x \rightarrow (y \rightarrow z) \vdash y \rightarrow (x \rightarrow z) \\
 & (x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) \\
 & b \rightarrow x \vdash \bar{b} \\
 & ((x \rightarrow y) \rightarrow x) \rightarrow x
 \end{aligned}$$

Note that the second and fourth of these are rules of inference; the fourth is stated imprecisely, meaning that the negation of a formula b follows from the claim that, for any x , if b , then x . Peirce leaves the propositional quantification here implicit, stating this rule only in English paraphrase. We could express his intentions more precisely by introducing a propositional constant for the false, and writing the axiom as $b \rightarrow \perp \vdash \bar{b}$. Peirce’s system is then a sound and complete axiomatization of a pure implicational propositional logic.

Peirce then introduces disjunction and conjunction, defining $x + y$ as $\bar{x} \rightarrow y$ and xy as $(\bar{x} + \bar{y})$. He introduces quantifiers by distinguishing “a pure Boolean expression referring to an individual and a Quantifying part saying what individual this is” (1885, 194). As a first approximation, he writes Any $(\bar{k} + h)$ to mean that any king in the universe of discourse is happy, and Some kh to mean that some king in that universe is happy. To handle relational terms, he introduces subscripts for what would now be considered individual variables, and uses Π and Σ to represent universal and existential quantification, respectively. So, the above would more precisely be written as $\Pi_i(\bar{k}_i + h_i)$ and $\Sigma_i k_i h_i$. He defines the quantifiers in terms of (potentially) infinite conjunctions and disjunctions, but notes that they are “only similar” to them, because the universe of

discourse may be uncountable (195). Peirce quickly shows off the power his new notation, writing $\Pi_i \Sigma_j l_{ij} b_{ij}$ to represent “everything is a lover and benefactor of something,” $\Pi_i \Sigma_j l_{ij} b_{ji}$ to represent “everything is a lover of a benefactor of itself,” $\Sigma_i \Pi_j (g_i l_{ij} + \bar{c}_j)$ to represent “if there be any chimeras there is some griffin that loves them all,” and $\Pi_j \Sigma_i (g_i l_{ij} + \bar{c}_j)$ to represent “each chimera (if there is any) is loved by some griffin or other” (195).

Peirce’s proof method requires converting formulas to prenex normal form, something that, he shows, preserves equivalence, and then using a technique similar to semantic tableaux. I will not go into details here. But note the dramatic shift from earlier conceptions of logic. Peirce formulates both axioms and rules—though not drawing attention to the difference between them—develops a logic of quantification, and then formulates a proof system. All of this advances significantly beyond the earlier algebraic approaches to logic, not to mention the rest of the logical tradition.

4.3 Frege’s *Begriffsschrift*

Gottlob Frege (1848–1925) published his *Begriffsschrift*, “ideography” or “concept writing,” in 1879. It has been called “perhaps the most important single work ever written in logic” (van Heijenoort 1967, 1). Frege’s system forms the basis for modern predicate logic. The twentieth-century achievements of Peano, Russell, Whitehead, Gödel, Tarski, Church, Carnap, and many others would have been impossible, or at least would have had a rather different character, without him. Frege’s goal, he wrote (1882), was to devise “a *lingua characterica* in Leibniz’s sense”—a universal language that, in Leibniz’s words, “depicts not words, but thoughts” (PW 13, NS 14)—the structure of which reflects not surface forms but underlying semantic structure, which Frege refers to as “conceptual content” (12).

Frege’s system parallels Peirce’s, not only in introducing quantifiers but also in employing, explicitly, the material conditional. Like Peirce’s system, it rests on the shoulders of Boole, DeMorgan, and others. But he makes a number of vital contributions. First, he rests his logic not on the distinction between propositions and terms or between subject and predicate but instead on the distinction between function and argument.

If in an expression, whose content need not be capable of becoming a judgment, a simple or a compound sign has one or more occurrences and if we regard that sign as replaceable in all or some of these occurrences by something else (but everywhere by the same thing), then we call that part that remains invariant in the expression a function, and the replaceable part the argument of the function. (1879, 22)

Second, he presents his system purely formally; for the first time, logic concerns formulas and their manipulation according to strictly syntactic operations. Third, he correspondingly draws a sharp distinction between axioms and rules of inference, with *modus ponens* and substitution as the only inference rules. Finally, he presents for the first time a complete set of rules for quantifiers, formulating universal instantiation and universal generalization as well as rules for identity. Frege is the first to see a quantifier as a variable-binding operator. He offers the first system in the history of logic that

unites propositional and predicate logic into a single unified system capable of handling nested quantifiers. He also presents the first higher-order logic. If modern logic begins in 1847, modern quantification theory begins in 1879.

Frege employs a notation that is difficult to reproduce, using strokes with concavities filled by German letters as universal quantifiers. Giuseppe Peano (1858–1932), an Italian mathematician, deserves credit for putting quantification theory into something close to its modern notation (Peano 1889), and I will use the Peano-inspired notation here. Frege’s account of universal quantification is this:

In the expression of a judgment we can always regard the combination of signs to the right of \vdash [the assertion stroke] as a function of one of the signs occurring in it. *If we replace this argument by a German letter and if in the content stroke we introduce a concavity with this German letter in it, ... this stands for the judgment that, whatever we may take for its argument, the function is a fact.* (1879, 24)

Suppose, in other words, we have the assertion that $p: \vdash p$. We can analyze p into function and argument: $\vdash \Phi(a)$. We can then replace the argument sign with a variable (German letters in Frege, Roman letters from the end of the alphabet, here), and prefix a quantifier on that variable, to obtain $\vdash \forall x\Phi(x)$. Frege explains the meaning of this kind of expression in terms of a rule of universal instantiation:

From such a judgment, therefore, we can always derive an arbitrary number of *judgments of less general content* by substituting each time something else for the German letter and then removing the concavity in the content stroke. (1879, 24)³⁶

From $\vdash \forall x\Phi(x)$, that is, we can derive $\vdash \Phi(a)$, $\vdash \Phi(b)$, and so on. Frege discusses scope explicitly (though he does not give a formal definition of it), and cautions that we cannot apply this rule when a universal quantifier occurs within the scope of an arbitrary operator. He thus restricts the rule to universal quantifiers that have as their scope the entire formula.

Frege formulates the rule of universal generalization: “A Latin letter may always be replaced by a German one that does not yet occur in the judgment” (1879, 25). Thus, given $\vdash \Phi(a)$, we may infer $\vdash \forall x\Phi(x)$, provided that x does not occur in Φ . This, together with his propositional rules and universal instantiation, gives him a complete set of rules for predicate logic.

Frege also introduces a derivable rule allowing the deduction from $\vdash \Phi(a) \rightarrow A$ to $\vdash \forall x\Phi(x) \rightarrow A$, provided that a does not occur in A and is replaced in every occurrence by x . He distinguishes $\neg\forall x\Phi(x)$ from $\forall x\neg\Phi(x)$, and defines “There are Φ ” as $\neg\forall x\neg\Phi(x)$. Frege shows how to express categorical propositions in his system, and constructs the square of opposition, without remarking that he has actually given up all relations except contradictories. His representations:

³⁶Frege never gives an independent semantic characterization of what universal quantification means. Taking this rule as giving such a characterization yields a substitutional conception of quantification. Frege, unlike Peirce, does not discuss the universe of discourse, and, to the extent that he has the concept of a universe at all, treats it as something given and static, not as something capable of varying from context to context, as in Peirce, or from model to model, as in contemporary model theory.

All S are P : $\forall x(S(x) \rightarrow P(x))$
 No S are P : $\forall x(S(x) \rightarrow \neg P(x))$
 Some S are P : $\neg \forall x(S(x) \rightarrow \neg P(x))$
 Some S are not P : $\neg \forall x(S(x) \rightarrow P(x))$

The notation makes it obvious that A and O propositions are contradictories, as are E and I propositions. A and E propositions, however, are not truly contraries, for they can be true together if there are no S s. Under just those circumstances, I and O propositions can also be false together, so they are not true subcontraries. Finally, A and E propositions do not entail I and O propositions, respectively, for the former lack the existential import of the latter.

Frege develops his system from nine axioms, together with rules of *modus ponens* and substitution. The axioms:

1. $a \rightarrow (b \rightarrow a)$
2. $(c \rightarrow (b \rightarrow a)) \rightarrow ((c \rightarrow b) \rightarrow (c \rightarrow a))$
3. $(d \rightarrow (b \rightarrow a)) \rightarrow (b \rightarrow (d \rightarrow a))$
4. $(b \rightarrow a) \rightarrow (\neg a \rightarrow \neg b)$
5. $\neg \neg a \rightarrow a$
6. $a \rightarrow \neg \neg a$
7. $c \equiv d \rightarrow (f(c) \rightarrow f(d))$
8. $c \equiv c$
9. $\forall x f(x) \rightarrow f(c)$

Frege claims completeness, but does not formulate the notion precisely, let alone offer an argument. Indeed, he cannot formulate it precisely, for he lacks the concept of an interpretation, and so has no precise concept of validity (Goldfarb 1979). Lukasiewicz (1934) shows that the first six are nevertheless a complete axiom system for propositional logic, and, indeed, that (3) is superfluous.

First-order logic makes a huge advance over Aristotelian logic in thinking about quantifiers as variable-binding expressions and thus in having the capacity to handle relational predicates. Aristotle himself realized that his theory did not incorporate inferences such as

Every perceiving is a thinking
 \therefore Every object of perception is an object of thought.

In the fourteenth century, and again in the seventeenth and nineteenth centuries, logicians returned to those worries, considering such inferences as

Every man's donkey is running.
 Every king is a man
 \therefore Every king's donkey is running. (Buridan 2001)

Every circle is a figure.
∴ Whoever draws a circle draws a figure. (Junge 1638)

All horses are animals
∴ Every horse's tail is an animal's tail. (DeMorgan 1847)

First-order logic, by incorporating relational predicates, solves these problems, unifying propositional and quantificational logic in a system powerful enough to express propositions and inferences that had escaped each on its own.

Frege's system is very powerful—in fact, as Russell later observed, *too* powerful—for it permits quantification over predicates as well as individual argument positions. It is thus tantamount to a full, impredicative second-order logic. Russell's (1902) letter to Frege praises the *Begriffsschrift* and the *Grundgesetze der Arithmetik*, which develops the system further, reducing the set of basic rules and axioms. But then he drops the bomb, known as Russell's paradox:

There is just one point where I have encountered a difficulty. You state [23] that a function, too, can act as the indeterminate element. This I formerly believed, but now this view seems doubtful to me because of the following contradiction. Let w be the predicate: to be a predicate that cannot be predicated of itself. Can w be predicated by itself? From each answer its opposite follows. Therefore we must conclude that w is not a predicate. Likewise there is no class (as a totality) of those classes which, each taken as a totality, do not belong to themselves. From this I conclude that under certain circumstances a definable collection [*Menge*] does not form a totality. (1902, 124–125)

Frege's response expresses consternation, but also the hope that Russell's discovery “will perhaps result in a great advance in logic, unwelcome as it may seem at first glance” (1902, 128). He nods in the direction of type theory:

Incidentally, it seems to me that the expression “a predicate is predicated of itself” is not exact. A predicate is as a rule a first-level function, and this function requires an object as argument and cannot have itself as argument (subject). Therefore I would prefer to say “a concept is predicated of its own extension”. (1902, 128)

In itself, however, that provides no solution. Russell's paradox cuts at the heart of Frege's reconstruction of arithmetic from logic. It does not, however, strike at Frege's system viewed as a system of first-order predicate logic. Peano, Russell, Whitehead, and others would simplify the unwieldy notation of the *Begriffsschrift*, turning first-order predicate logic into the *lingua franca* of twentieth-century philosophy. Indeed, so influential would Frege's system become that philosophers as prominent as W. V. O. Quine (1960), Donald Davidson (1967), and John Wallace (1970) would write of it as “*the* frame of reference,” the logic into which all assertions must be translated if they are to be considered meaningful at all.

5 Contemporary Quantification Theory

In later work, especially Frege (1892), Frege offers two ideas that have led to contemporary approaches to quantification. First, he makes explicit the Aristotelian thought that determiners such as ‘some’ and ‘all’ are relations—in Frege’s language, relations between concepts:

We may say in brief, taking ‘subject’ and ‘predicate’ in the linguistic sense: A concept is the reference of a predicate; An object is something that can never be the whole reference of a predicate, but can be the reference of a subject. It must here be remarked that the words ‘all’, ‘any’, ‘no’, ‘some’, are prefixed to concept-words. In universal and particular affirmative and negative sentences, we are expressing relations between concepts; we use these words to indicate the special kind of relation. They are thus, logically speaking, not to be more closely associated with the concept-words that follow them, but are to be related to the sentence as a whole. (Frege 1892, 1951, 173)

Second, Frege treats quantifiers as second-level concepts. “I have called existence a property of a concept,” he writes (1892, 174); he uses the same analysis for quantifiers. These two ideas have led to the contemporary theory of generalized quantifiers. It took more than sixty years, however, for that generalization to be developed, and almost a century before its significance began to be appreciated.

5.1 Limitations of Classical First-Order Logic

The theory of quantification that issued from the work of Peirce and Frege became known as classical first-order logic. The early part of the twentieth century witnessed the discovery of many of its deepest properties through the development of metalogic, especially in the form of model theory and recursion theory. Some highlights:

1. Leopold Löwenheim (1915) and Thoralf Skolem (1920), starting from the work of Peirce and Schröder, show that, in Skolem’s formulation, “Every proposition in normal form either is a contradiction or is already satisfiable in a finite or countably infinite domain” (Skolem 1920, 256). Since every proposition is equivalent to one in normal form, this shows that every satisfiable proposition is satisfiable in a countable domain. He extends this to infinite sets of formulas, and shows that every uncountable domain satisfying a set of formulas has a countable subset also satisfying that set. Alfred Tarski and Carl Vaught (1956) later prove an upward form of this theorem, showing that any set satisfiable in an infinite domain is satisfiable in an uncountable domain.
2. Skolem (1928) shows that every formula of first-order logic can be “Skolemized” by finding an equivalent in prenex normal form, dropping all quantifiers, and then replacing existentially quantified variables with function terms containing variables bound universally to the left of that existential quantifier. Thus, a Skolemized version of $\forall x\forall y\exists z(Fxy \rightarrow (Fxz \wedge Fyz))$ would be $Fxy \rightarrow (Fxf(xy) \wedge$

$Fy f(xy)$). He uses this to develop a non-axiomatic proof method for first-order logic that amounts to a decision procedure for determining, for a given formula A and natural number n , whether A is satisfiable in a domain of cardinality n .

3. Kurt Gödel (1930) proves the completeness theorem, showing that every valid formula is provable. Specifically, Gödel shows that every formula “*is either refutable or satisfiable* (and, moreover, satisfiable in the denumerable domain of individuals)” (1930, 585). He deduces from it the compactness theorem, showing that a set of formulas is satisfiable if and only if every finite subset of it is satisfiable. The Löwenheim-Skolem theorem is of course also a consequence.
4. Alfred Tarski (1936), in his investigations of the concept of truth, constructs a formal semantics giving truth conditions for quantified sentences and thus lays the foundation for model theory. Earlier thinkers had given informal characterizations of the truth conditions for quantified sentences, linking them to generalized conjunction or disjunction and perhaps to a universe of discourse. Tarski gives the first formal characterization of the semantics of quantified sentences, counting $\forall x\Phi(x)$ true on a model \mathfrak{M} , relative to an assignment g of objects from \mathfrak{M} 's domain to variables, if and only if $\Phi(x)$ is true on every assignment g' differing from g solely in its assignment to x . The relativization to an assignment function disappears for closed sentences, i.e., those without free variables. Tarski also shows that, although the concept of truth can be given a consistent and precise characterization in the metalanguage, it cannot be given a similar characterization in the object language.
5. Alonzo Church (1936) proves that first-order logic is undecidable. Since propositional logic is decidable, this shows that Boole's idea that a single abstract theory might be at once a theory of propositional relations (on one interpretation) and a theory of quantification (on another interpretation) is hopeless.
6. Per Lindström (1969, 1974) proves that classical first-order logic is the strongest compact logic having the downward Löwenheim-Skolem property, developing the foundations of abstract model theory.

These results represent impressive intellectual achievements. They also point to some limitations of classical first-order logic. Löwenheim and Skolem's result, for example, shows that first-order logic cannot distinguish countable from uncountable domains. ‘There are countably many’ and ‘there are uncountably many’ thus have no first-order representations. It is not hard to show that ‘there are finitely many’ and ‘there are infinitely many’ similarly have no first-order representations.

Subsequent developments point to additional limitations. Peter Geach, by way of his discussions of medieval (specifically, fourteenth-century) logic, points out many of the most important. First, he draws attention to determiners beyond those analyzed by earlier logical systems. Remarkably, for more than two thousand years, the study of quantification was essentially the study of ‘every,’ ‘no,’ and ‘some,’ together with determiners taken as equivalent or reducible to them (‘all,’ ‘never,’ etc.). De Morgan (1847, 1860) broke new ground in thinking about determiners such as ‘most,’ noting the validity of ‘Most S s are P s; most S s are P 's; \therefore some P s are P 's.’ De Morgan also

thought about numerical quantifiers, noting the validity, where s is the number of Ys and $n + m > s$, of ‘ n Ys are Xs ; m Ys are Zs ; $\therefore n + m - s$ Xs are Zs .’ Subsequent mathematical logicians nevertheless ignored such quantifiers, and indeed developed systems in which ‘most’ could not receive a treatment analogous to that accorded ‘some’ and ‘all.’ Nicholas Rescher (1964) showed that ‘most,’ like ‘there are finitely many,’ ‘there are infinitely many,’ ‘there are uncountably many,’ and the like, has no first-order representation. ‘Most’ poses a puzzle for Aristotelian logic in its nineteenth-century formulation, according to which every term must be distributed or undistributed, for ‘most’ fits neither pattern. But it also poses a puzzle for first-order logic, which lacks the resources to define it or even introduce it on analogy with \forall or \exists . David Lewis 1975 similarly observes the wide range of quantifiers available in English, few of which find representation in first-order logic.

Second, Geach draws attention to sentences such as this, known as the Geach-Kaplan sentence: ‘Some critics admire only one another.’ ‘Some critics admire only each other’ has a reading expressible in first-order logic—as $\exists x \exists y ((Cx \wedge Cy) \wedge \forall z (Axz \rightarrow z = y) \wedge \forall w (Ayw \rightarrow w = x))$ —but the intended reading of the ‘one another’ version, which encompasses sets of mutually admiring critics of any cardinality, does not. It requires quantification over sets: $\exists X \forall x (x \in X \rightarrow (Cx \wedge \forall y (Axy \rightarrow y \in X)))$.

Third, Geach draws attention to so-called “donkey sentences,” such as ‘Every farmer who owns a donkey beats it,’ in which there is an anaphoric connection between subject and predicate. The difficulty is not in finding a first-order expression with the appropriate truth conditions— $\forall x \forall y ((Fx \wedge Dy \wedge Oxy) \rightarrow Bxy)$ serves adequately—but in reaching that representation in a compositional fashion. Symbolizing ‘Every farmer owns a donkey’ requires an existential quantifier: $\forall x (Fx \rightarrow \exists y (Dy \wedge Oxy))$. Where there is no anaphoric connection between subject and predicate, that works fine; we can represent ‘Every farmer who owns a donkey is happy’ as $\forall x ((Fx \wedge \exists y (Dy \wedge Oxy)) \rightarrow Hx)$. Pursuing that strategy for ‘Every farmer who owns a donkey beats it,’ however, would produce $\forall x ((Fx \wedge \exists y (Dy \wedge Oxy)) \rightarrow Bxy)$, which does not serve the purpose at all, since the final ‘ x ’ is not bound by any quantifier. Fourteenth-century logicians, as we have seen, worry about such sentences, recognizing that quantifier phrases not only contribute to truth conditions but also, under certain conditions, enable later anaphoric reference. Nothing in Aristotle’s theory could explain that function. Nothing in first-order logic explains it either.

Fourth, Geach draws attention to cases involving such anaphoric connection in which it seems impossible to express the desired connection without an unwanted existential commitment. Consider

Hob believes that a witch blighted Bob’s mare, and Nob believes that she blighted Cob’s cow.

The existential expression in the first phrase occurs within a belief context; representing it as something like

Hob believes that $[\exists x (Wx \wedge Bxm(b))]$

makes no commitment to the existence of witches. Hob believes there is a witch, but, to represent what the sentence says, we do not have to believe it. The anaphoric

pronoun ‘she’ in the second phrase, however, creates a problem similar to that in donkey sentences. Appending another clause containing the variable x leaves it unbound; quantifying it existentially within Nob’s belief context conveys that Hob believes some witch has blighted Bob’s mare, and that Nob believes that some witch has blighted Cob’s cow, but does nothing to convey that Nob believes them to be the same witch. Existentially quantifying outside the belief contexts allows us to capture that, but at the cost of making the sentence commit us to the existence of witches. But surely one can report on the beliefs of Hob and Nob without oneself expressing belief in witches.

Geach’s Hob–Nob example is one case in which first-order logic seems to require unwanted existential assumptions. There are many others: ‘I want a sloop’ (Quine 1956), which threatens to commit us to the existence of a particular sloop that I want, or ‘John is seeking a unicorn’ (Montague 1974), which threatens to commit us to unicorns. Such concerns have led to the development of free logic (logic free of existential commitments with respect to its singular terms) and to Montague grammar.

Finally, Leon Henkin (1961), Jaacko Hintikka (1979), and Jon Barwise (1979) draw attention to branching quantifiers, which seem to defy not only the bounds of first-order logic but any system of linear representation. Consider a sentence such as ‘A leader from each tribe and a captain from each regiment met at the peace conference.’ Symbolizing this as something beginning with $\forall x \exists y \forall z \exists u$ fails to capture the intended meaning for reasons that become obvious when one thinks about Skolemization. A Skolemized version of a linear representation would replace y with $f(x)$ and u with $g(x, y, z)$ rather than the intended $g(z)$. The choice of leader should depend solely on the tribe; the choice of captain should depend solely on the regiment. A linear representation, however, makes the choice of a value for u depend not only on the value of z but on the values of x , y , and z .

These problems, taken together, suggest that first-order logic does not analyze quantification in a manner adequate to understanding natural language quantification. Though it remains the logic of choice for understanding quantification in mathematics, doubt has begun to arise about its adequacy even there. First-order arithmetic is not categorical; it has non-standard models, with domains consisting not only of the natural numbers but also of one or more \mathbb{Z} -chains, that is, chains with the structure of the integers. Second-order arithmetic, in contrast, is categorical. That suggests to many that first-order logic is not adequate even to mathematical reasoning.

5.2 Generalized Quantifiers

First-order logic gains its power, in part, by moving from a monadic conception of predicates to a relational one. It diverges from Aristotelian logic, however, by adopting a monadic theory of quantification. A quantifier $\forall x$ or $\exists x$ operates on a single formula, however complex. In Frege’s language, quantifiers are second-level concepts; they act as monadic predicates of monadic predicates—in effect, as terms applying to terms.

This has the effect of making quantifier expressions “disappear” in symbolic representations. In representing ‘Every king’s donkey is running,’ for example, we lose the ability to identify any particular component as corresponding to ‘every king’s donkey’; the formula $\forall x \forall y ((Kx \wedge Dy \wedge Oxy) \rightarrow Ry)$ has no subformula that represents just that noun phrase. More dramatic is the contrast between ‘Every farmer who owns

a donkey is happy' ($\forall x((Fx \wedge \exists y(Dy \wedge Oxy)) \rightarrow Hx)$) and 'Every farmer who owns a donkey beats it' ($\forall x\forall y((Fx \wedge Dy \wedge Oxy) \rightarrow Bxy)$). This leads Frege and Russell to stress a context principle, that only in the context of a sentence does an expression have meaning. The surface form of a sentence, they hold, is misleading; the true structure of the sentence emerges only at the level of logical form. The distinction between subject and predicate on which logic before the middle of the nineteenth century relies appears, from this point of view, as a misleading artifact of the inadequacies of natural language. What Frege and Russell take as a deep insight, however, might instead be viewed as a defect of their theory.

Suppose we were to combine a relational conception of predicates with a relational theory of quantifiers, seeing determiners as expressing relations between terms or their extensions. The result is the theory of generalized quantifiers, developed in Mostowski (1957), Lindström (1966), Barwise and Cooper (1981), Barwise and Feferman (1985), and many subsequent works. This theory has several advantages over first-order logic as a theory of quantification, including a full range of determiners, handling problem sentences, and respecting compositionality, giving the concepts of subject and predicate a respectable role to play.

We may endorse Frege's thought that quantifiers are second-level concepts and his thought that determiners express relations among concepts without limiting ourselves to quantifiers of Fregean form. *Determiners* are expressions such as 'every,' 'all,' 'some,' 'no,' 'most,' 'at least n ,' 'finitely many,' 'uncountably many,' and the like; they denote *quantifiers*. Noun phrases are expressions such as 'every farmer,' 'all donkeys,' 'some European kings of the fourteenth century,' 'no wives of Henry VIII,' 'most donkeys owned by a farmer,' 'at least n students,' 'finitely many numbers,' 'uncountably many functions from numbers to numbers,' and the like. Definite and indefinite articles may be included among the determiners; proper names may be included among the noun phrases.

Let's start, as Aristotle does, with categorical propositions. 'Every S is P ,' 'Some S is P ,' and so on have the form

Determiner Subject Predicate

We can think of determiners as relations between terms, in effect construing this as

Determiner (Subject, Predicate)

Or, we can see the determiner and subject as forming a noun phrase, and then applying to the predicate:

(Determiner Subject)(Predicate)

The former is more intuitive and closer to most of the logical tradition. Either one, however, can serve as the basis for a theory of generalized quantifiers. Barwise and Cooper 1981, Keenan and Moss 1984, and Keenan and Stavi 1986 adopt the latter strategy; van Benthem 1983, 1984, 1986, 1987, 1989, and westerståhl 1989 adopt the former.³⁷

³⁷Good summaries of the theory of generalized quantifiers include van der Does and van Eijck 1996, Keenan 1996, Peters and westerståhl 2006, and Keenan and westerståhl 1997, 2011.

Letting U represent a universe of discourse, we can think of a categorical proposition, then, as having the structure

$$D_U S P$$

If we think of quantification as extensional, we can think of terms as standing simply for their extensions, and so think of a quantifier, relative to a universe of discourse, as a relation between sets. The standard examples of quantifiers are binary relations, having the above structure, but there are quantifiers such as ‘more S s than S ’s’ that relate more than two sets. So, generally, we can think of a quantifier as an n -ary relation on sets. Since U is the universe of discourse, we are concerned only with the extensions of S , P , and any other terms involved in this relation within that universe. So, we may see a quantifier Q on U as an n -ary relation on subsets of U : $Q_U \subseteq (\wp(U))^n$.

Not every n -ary relation on subsets of a universe of discourse, however, is a quantifier. Quantifiers satisfy a number of constraints that Barwise and Cooper refer to as natural language universals. For the moment, let’s restrict our attention to binary quantifiers.

First, quantifiers satisfy a principle of *Extension*:

$$Q_U S P \wedge U \subseteq U' \Rightarrow Q_{U'} S P$$

The universe of discourse can expand without affecting the truth of a quantified sentence, provided that the expansion has no effect on the extensions of the related terms.

Second, quantifiers *live on* their sets, in the sense that satisfy a principle of *conservativity*:

$$Q_U S P \Leftrightarrow Q_U S (S \cap P)$$

‘Every S is P ’ is equivalent to ‘Every S is S -and- P ’: ‘Every man is mortal’ is equivalent to ‘every man is a mortal man.’ As westerståhl (1989) notes, this distinguishes subjects from predicates, giving subjects “a privileged role” (38), for all that matters to the truth of a quantified sentence are the properties of subsets of the subject’s extension. These constraints, together, allow us to drop relativization to the universe of discourse. Conservativity guarantees that we can, without loss of generality, view the predicate’s extension as a subset of the subject’s extension. Extension guarantees that no universe of discourse larger than the subject’s extension affects truth values. So, we can identify the universe with the extension of the subject:

$$S \subseteq U \Rightarrow (Q_U S P \Leftrightarrow Q_S S P)$$

Hereafter, I drop relativization to the universe of discourse and write $Q S P$ whenever, in context, the subject term remains constant.

As the term ‘quantifier’ suggests, the truth value of a quantified proposition depends solely on quantities, not on other aspects of the sets concerned. One way to capture this idea is to say that the truth of a quantified proposition is invariant under permutations. Permute the universe of discourse, so that related subsets change while retaining their cardinalities, and truth values of quantified propositions should remain unaltered. Quantifiers thus satisfy a principle of *permutation* or *isomorphism*: If f is a one-to-one correspondence from U onto U' , then

$$Q_U S P \Leftrightarrow Q_{U'} f[S] f[P]$$

This has the effect that the truth values of quantified propositions depend only on the cardinalities of the sets concerned. So, we can view a binary quantifier as a relation between $|S|$ and $|S \cap P|$, or, equivalently, between $|S - P|$ and $|S \cap P|$.

Here, for example, is how we might express some common quantifiers as relations between sets and as relations between cardinalities of sets:

$$\begin{aligned}
\text{Every } S \text{ is } P &\Leftrightarrow S \subseteq P \Leftrightarrow |S| = |S \cap P| \\
\text{Some } S \text{ is } P &\Leftrightarrow S \cap P \neq \emptyset \Leftrightarrow |S \cap P| > 0 \\
\text{No } S \text{ is } P &\Leftrightarrow S \cap P = \emptyset \Leftrightarrow |S \cap P| = 0^{38} \\
\text{Most } S \text{ are } P &\Leftrightarrow |S \cap P| > |S - P|^{39} \\
\text{At least } n \text{ } S \text{ are } P &\Leftrightarrow |S \cap P| > n \\
\text{At most } n \text{ } S \text{ are } P &\Leftrightarrow |S \cap P| \leq n \\
\text{Exactly } n \text{ } S \text{ are } P &\Leftrightarrow |S \cap P| = n \\
\text{Finitely many } S \text{ are } P &\Leftrightarrow |S \cap P| \text{ is finite} \\
\text{Uncountably many } S \text{ are } P &\Leftrightarrow |S \cap P| \text{ is uncountable} \\
\text{All but finitely many } S \text{ are } P &\Leftrightarrow |S - P| \text{ is finite} \\
\text{All but (exactly) } n \text{ } S \text{ are } P &\Leftrightarrow |S - P| = n
\end{aligned}$$

We can do the same for quantifiers that are not binary:

$$\begin{aligned}
\text{More } S \text{ than } S' \text{ are } P &\Leftrightarrow |S \cap P| > |S' \cap P| \\
\text{At least as many } S \text{ as } S' \text{ are } P &\Leftrightarrow |S \cap P| \geq |S' \cap P| \\
\text{Exactly as many } S \text{ as } S' \text{ are } P &\Leftrightarrow |S \cap P| = |S' \cap P|
\end{aligned}$$

There are of course many additional quantifiers in natural language, including ‘several,’ ‘many,’ ‘few,’ ‘a few,’ ‘the,’ ‘this,’ ‘that,’ ‘these,’ ‘those,’ ‘one,’ ‘both,’ and ‘neither.’

Since binary quantifiers are binary relations, we can classify them according to properties as we can other relations. A quantifier Q is reflexive, for example, iff QSS , and symmetric iff $QSP \Leftrightarrow QPS$. ‘All’ is reflexive; ‘some’ and ‘no’ are symmetric. (Aristotle expressed the same thought by saying that the latter convert simply.) Keenan (1987) generalizes symmetry to a property he identifies with weakness, that is, acceptability in ‘There is’ or ‘there are’ contexts. Q is *intersective* iff Q is conservative and $QS_1 \dots S_n P \Leftrightarrow Q(S_1 \cap P) \dots (S_n \cap P)P$. Binary quantifiers are symmetric iff they are intersective (van der Does and van Eijck 1996, 11).

Barwise and Cooper generalize transitivity to persistence and monotonicity, sometimes called left- and right-monotonicity:

$$\begin{aligned}
Q \text{ is mon}\uparrow &\text{ iff } QSP \wedge P \subseteq P' \Rightarrow QS P' \\
Q \text{ is mon}\downarrow &\text{ iff } QSP \wedge P' \subseteq P \Rightarrow QS P' \\
Q \text{ is }\uparrow\text{mon} &\text{ iff } QSP \wedge S \subseteq S' \Rightarrow QS' P \\
Q \text{ is }\downarrow\text{mon} &\text{ iff } QSP \wedge S' \subseteq S \Rightarrow QS' P
\end{aligned}$$

³⁸This explains why these categorical statement forms could be expressed in Boolean notation as $x = xy$, $xy \neq 0$, and $xy = 0$.

³⁹This assumes that ‘most’ is equivalent to ‘more than half.’ In fact, it is probably stronger; giving precise truth conditions is difficult. See Kennan 1996, Kennan and westerstahl 1997, Ariel 2003, 2004, 2006, Pietroski, Lidz, Hunter, and Halberda 2009, and Hackl 2009.

These properties are responsible for the validity and invalidity of syllogistic inferences. We can summarize the properties of Aristotelian quantifiers straightforwardly:

Every: reflexive, $\downarrow\text{mon}\uparrow$
 Some: intersective, $\uparrow\text{mon}\uparrow$
 No: intersective, $\downarrow\text{mon}\downarrow$

If we were to associate the left and right arrows with the subject and predicate terms in a categorical proposition, then a downward arrow would correspond to the term's being distributed, and an upward arrow would correspond to its being undistributed. This explains why the medieval concept of distribution could be used to developed rules for syllogistic validity. It also explains why nineteenth century logicians who think that a term must be either distributed or undistributed are making a mistake. Consider the properties of a wider class of quantifiers:

Most: $\text{mon}\uparrow$
 At least n : intersective, $\uparrow\text{mon}\uparrow$
 At most n : intersective, $\downarrow\text{mon}\downarrow$
 Exactly n : intersective
 Finitely many: intersective, $\downarrow\text{mon}\downarrow$
 Uncountably many: intersective, $\uparrow\text{mon}\uparrow$
 All but finitely many: $\downarrow\text{mon}\uparrow$
 All but exactly n :

In 'Most S are P ,' S is neither distributed nor undistributed. The same is true of S in 'Exactly one S is P ' and 'All but exactly two S are P .' Logicians working in the Aristotelian tradition could easily have expanded their theories to account for determiners such as 'at least n ,' 'several,' 'a few,' 'uncountably many'—which would pattern as particular quantifiers—and 'at most n ,' 'finitely many,' and 'few'—which would pattern as universal negatives—but did not have the conceptual tools to incorporate nonmonotonic determiners such as 'most' and 'exactly n .'

We have so far been thinking of a quantifier Q on U as an n -ary relation on subsets of U : $Q_U \subseteq (\wp(U))^n$. We can generalize this further to n -ary relations among relations on U . A *local quantifier of type* $\langle k_1, \dots, k_n \rangle$ on U is an n -ary relation between subsets of U^{k_1}, \dots, U^{k_n} : $Q_U \subseteq \wp(U^{k_1}) \times \dots \times \wp(U^{k_n})$. A *global quantifier of type* $\langle k_1, \dots, k_n \rangle$ is a map from universes U to local quantifier of type $\langle k_1, \dots, k_n \rangle$ on U . This generalization permits us to introduce a *Henkin* or *branching quantifier* H of type $\langle 4 \rangle$, such that $H = \{R \subseteq U^4 : \exists f, g \subseteq U^2 \forall x, z \in U \langle x, f(x), z, g(z) \rangle \in R\}$. We can then express a branching quantifier sentence such as 'A leader from every tribe and a captain from every regiment met at the peace conference' as $HxyzMxyzu$.⁴⁰

The relational conception of quantifiers developed here is equivalent to the conception of Montague (1974) and Barwise and Cooper (1981) according to which a quantifier is a function from subsets of the universe of discourse to sets of subsets, for we may think of a quantifier as mapping a set into the set of all sets to which it relates that

⁴⁰For a detailed treatment, see Sher 1997.

set: $Q(S) = \{P : QSP\}$. The Montagovian conception, though less intuitive, has the advantage of making it easy to compute semantic values compositionally. Determiners map common nouns into noun phrases. Quantifiers, correspondingly, map sets, the denotations of common nouns, into families of sets, the denotations of noun phrases. We might take a proper noun such as ‘Socrates’ not as standing for an individual but instead as standing for a family of sets, that is, the family of extensions of predicates that apply to Socrates. Similarly, we might take ‘every man’ not as standing for the set of all men, as some nineteenth-century logicians were tempted to do, but as standing for a family of sets, the set of extensions of predicates applying to every man.

5.3 Quantification and Anaphora

Fourteenth-century logicians such as Burley and Buridan note that quantifiers not only express relations between terms but also introduce items in discourse that can be referred to anaphorically. The problem this raises is not one of truth conditions, as we have seen—it is not difficult to give first-order representations of sentences such as ‘an animal is running and it is a man’ or ‘Every farmer who owns a donkey beats it’—but rather of compositionality. The problem, in other words, is typically not a lack of appropriate truth conditions but a rule-governed way of generating them.

In Aristotelian logic, we can represent ‘an animal is running’ as ‘Some A is R ,’ but we then have no way of indicating that the running animal is a man. Appending another categorical proposition will not help. In first-order logic, similarly, we can represent ‘an animal is running’ as $\exists x(Ax \wedge Rx)$. Having done that, however, we have no way of adding another formula representing that the x in question is a man. What we have to do in both cases is extend the original representation, writing ‘Some A is R and M ’ or $\exists x(Ax \wedge Rx \wedge Mx)$. Since the anaphora occurs within the same sentence, we could perhaps write rules requiring this. But similar anaphora can occur across sentential boundaries. Buridan’s example could just as easily have been ‘an animal is running. It is a man.’ These could moreover be separated by intervening discourse: ‘an animal is running. People look on in surprise. It is uncommon to see such a sight in the square, amidst the crowds, in the midday heat. It is a man.’ This makes a strategy of waiting to close off the representation of the first sentence until all later anaphors have been collected implausible.

Sometimes, the problem is a lack of appropriate truth conditions. Consider this discourse:

Mary: “A man fell over the edge!”
 John: “He didn’t fall; he jumped.”

A quantificational analysis leads to the formula $\exists x (x \text{ is a man} \wedge x \text{ fell over the edge} \wedge x \text{ didn't fall} \wedge x \text{ jumped})$ for the discourse, which is contradictory. This seems appropriate enough, for the second assertion contradicts the first. But we have no way of telling whether the second assertion, considered alone, is true or false, for it receives no independent truth conditions. Representing just John’s assertion would yield $\exists x (x \text{ didn't fall} \wedge x \text{ jumped})$, which is true if anything jumped and didn’t fall. Nothing there ties the assertion to the person spoken of by Mary.

The problem posed by so-called donkey sentences is thus quite general. Quantified noun phrases not only relate sets or relations on a universe of discourse but also make subsequent anaphoric connection to certain things or sets possible.

Lauri Karttunen (1976) hypothesizes that quantified noun phrases introduce *discourse referents*, items to which later anaphoric elements can link. Hans Kamp (1981) and Irene Heim (1982) develop that hypothesis into a theory, which has become known as Discourse Representation Theory (DRT). An appearance of a noun phrase (hereafter, NP) that is indefinite, that is, of the form $a(n) F$, establishes a discourse referent if and only if it justifies the occurrence of a coreferential pronoun or definite NP later in the text. For example, in example (1), the indefinite NP *a man* establishes a discourse referent. It justifies the coreferential pronoun *he* in the second sentence, in the sense that the second sentence cannot occur without the first in the absence of something else that would justify the occurrence of such a pronoun—a linguistic context providing other possibilities of anaphoric linkage, for example, or an act of demonstration such as pointing that would make the pronoun deictic rather than anaphoric.

Karttunen's notion, unlike most traditional syntactic and semantic concepts, is multisentential. It applies readily to anaphora across sentential boundaries. It is also procedural. Karttunen defines not what a discourse referent is but what it takes to establish one. He thus analyzes indefinite NPs not in terms of what they stand for but in terms of what they do. Dynamic semantics extends this approach to language in general. It sees a sentence as a way of transforming one context into another.

NPs such as 'an animal' do not, however, always establish discourse referents in Karttunen's sense of licensing further anaphora. In particular, when they occur within the scope of quantifiers or negations, they do not permit further anaphoric links:

Every farmer owns a donkey. *I feed it sometimes.
It is not true that an animal is running. *It's a man.

An adequate theory based on Karttunen's approach, then, should explain how and when NPs license anaphora. Discourse Representation Theory (DRT) does this in two steps. First, it analyzes indefinites and other NPs with $\uparrow\text{mon}\uparrow$ determiners—those fitting the pattern of Aristotle's particular quantifiers—as introducing discourse referents in every case. Only sometimes, however, are those referents accessible to later anaphors. Second, it specifies formally a relation of accessibility that determines when anaphoric connections are possible. That relation depends crucially on the Discourse Representation Structure built from the discourse.

Kamp (1981) presents the essentials of Discourse Representation Theory. The general form of the theory is to process a discourse in two steps. First, it parses a sentence syntactically and applies an algorithm to construct a Discourse Representation Structure (DRS). Second, DRSs receive truth conditions by way of a model-theoretic semantics.

The intermediate level of DRSs, or semantic representations, is the key to the theory's novelty. The DRS represents both context and content. Think of prior discourse, for example, as having built up an initial DRS. Then a sentence, processed by the DRS construction algorithm, transforms that initial DRS, acting as content of the previous discourse and context of the current sentence, into a new DRS, representing the content

of the entire discourse and the context of further utterances. At each stage of the construction, the DRS determines the truth conditions of the discourse up to that point. But it also constitutes a conceptual level that provides information that cannot be recovered from truth conditions alone.

In short, then, Discourse Representation Theory treats indefinites and other NPs with similar logical properties as referring expressions at the level of the DRS construction algorithm and as quantifiers at the level of the truth definition. Pronouns do not introduce independent elements into a DRS; they refer to items already there. They select their referents from sets of antecedently available entities. Deictic pronouns do so from the real world; anaphoric pronouns select from constituents of the representation—Karttunen’s discourse referents. The theory tries to specify sets of referential candidates, by specifying which entities are accessible to a given anaphor. Strategies for selecting referents from among the set of possible referents are complex, relying on semantic, pragmatic, and discourse factors. The theory itself does not spell out these strategies, but is easily supplemented with them.

The DRS construction algorithm provides rules for building or altering DRSs, given the syntactic parse of a sentence—specifically, an analyzed surface structure. To see how it works, let’s begin with ‘An animal is running.’ I will ignore tense, aspect, and other complications; the following DRS would be a component of a richer structure incorporating this additional information.

A DRS $m = \langle U, C \rangle$ consists of a *universe* or *domain* and a *set of conditions*. The universe is a set of entities to be thought of as discourse referents; the conditions provide information about those referents. (For convenience, I shall write a DRS $\langle \{x_1, \dots, x_n\}, \{C_1, \dots, C_m\} \rangle$ as $\langle x_1, \dots, x_n : C_1, \dots, C_m \rangle$.) One can thus think of a DRS as an information state or as a partial model, a representation of a situation or part of a world.

Indefinite NPs introduce discourse referents into the universe. They also introduce conditions—entries providing information about discourse referents—expressing the content of the description. ‘An animal,’ then, introduces a discourse referent and a condition saying that it is an animal. Names and personal pronouns also introduce discourse referents and, often, additional conditions.

Finally, verbs and adjectives introduce conditions. Applying the construction algorithm to *An animal is running*, in the null context, produces the following DRS: $\langle u : \text{animal}(u), u \text{ is running} \rangle$. This DRS consists of a domain $\{u\}$ and a set of conditions: $\{\text{animal}(u), u \text{ is running}\}$, which express or simply are properties and relations among objects in that domain. It represents the content of ‘An animal is running’ and acts as a context for ‘It’s a man.’ The anaphoric pronoun *it* must refer to a discourse referent already introduced. We might, as soon as we reach it in the construction algorithm, search for that referent and use it in the conditions to be introduced. To separate the problem of searching among the possible referents for the actual one, however, it is more convenient to introduce another discourse referent, together with a condition identifying it with a previously introduced referent. So, we obtain a DRS of this form: $\langle u, v : \text{animal}(u), u \text{ is running}, \text{man}(v), v = ? \rangle$. What are the possible referents of v ? The theory states that, in the absence of an act of demonstration, all candidates are discourse referents previously introduced. Assuming the earlier DRS to be the entire context for the utterance, nothing but u is available. Identifying v with u then yields

$\langle u, v : \text{animal}(u), u \text{ is running}, \text{man}(v), v = u \rangle$.

We have now set up enough machinery in discourse representation theory to handle simple cases of anaphoric connection. The theory correctly predicts the anaphoric properties of the sentences in Buridan's very simple discourse. After processing the first sentence, we obtain the first DRS above; after processing the second, we obtain the second DRS, which embodies the information conveyed by both sentences.

This situation is quite general. The construction algorithm operates on sentence 1 in context 0, producing DRS 1. It then takes DRS 1 as context in processing sentence 2, yielding DRS 2, and so on. At each stage, the DRS produced embodies the information in all sentences processed up to that stage. Indeed, it must; otherwise it could not serve as context for the next sentence. The meaning of each sentence is not a truth condition that can be stated independently of previous discourse but a function taking discourse contexts into discourse contexts.

Discourse Representation Theory provides a semantics for DRSs and, thereby, for sentences and discourses. A discourse is true if and only if the DRS built from it by the construction algorithm is true. So, primarily, we will speak of DRSs as having truth values.

A DRS is a partial model. It is true in a model \mathfrak{M} if and only if it is a part of \mathfrak{M} ; there must be a way of embedding the DRS into \mathfrak{M} . More formally, a DRS m is true in a model \mathfrak{M} if and only if there is a homomorphic embedding of m into \mathfrak{M} . This means that there must be a function from the universe of m into that of \mathfrak{M} preserving the properties and relations m specifies. More formally still, there must be a function $f: U_m \rightarrow U_{\mathfrak{M}}$ such that, if $\langle a_1, \dots, a_n \rangle \in F_m(R)$, then $\langle f(a_1), \dots, f(a_n) \rangle \in F_{\mathfrak{M}}(R)$.

To see how the theory accounts for the quantificational force of indefinites, consider the DRS above, generated by the construction algorithm from the sentence 'An animal is running.' This DRS consists of a domain $\{u\}$ and a set of conditions $\{\text{animal}(u), u \text{ is running}\}$. The DRS is true in a model \mathfrak{M} , according to our definition, if and only if it can be embedded into \mathfrak{M} . This means that there must be a function f from $\{u\}$ into $U_{\mathfrak{M}}$ such that $f(u) \in F_{\mathfrak{M}}(\text{animal})$ and $f(u) \in F_{\mathfrak{M}}(\text{is running})$. Thus, the DRS, and the sentence that generated it, are true if and only if some animal is running, exactly as we would expect.

The DRS the construction algorithm generates from the entire discourse is true in a model \mathfrak{M} if and only if there is a function f from $\{u, v\}$ into $U_{\mathfrak{M}}$ such that $f(u) \in F_{\mathfrak{M}}(\text{animal})$, $f(u) \in F_{\mathfrak{M}}(\text{is running})$, $f(v) \in F_{\mathfrak{M}}(\text{man})$, and $f(u) = f(v)$. The DRS is true, in other words, if there is an animal that is running and is also a man.

Note that the indefinite *an animal* has quantificational force here, in the sense that the truth conditions for the sentence might appropriately be represented by an existentially quantified formula, even though the indefinite did not introduce a quantifier into the DRS. It introduced nothing but a discourse referent and a condition. In Discourse Representation Theory, indefinite descriptions are referential terms, not existential quantifiers. There is nevertheless no simple answer to the question of what they denote. Their contribution to truth conditions depends on the role played by the clause containing the description, which depends, in turn, on the structure of the DRS. The same is true of any NP with a $\uparrow\text{mon}\uparrow$ determiner. The theory accounts not only for the quantificational force of such NPs but also for their frequent success in establishing discourse referents, licensing further anaphoric connections.

Discourse Representation Theory thus explains the quantificational and anaphoric characteristics of the indefinite *an animal* at different levels of the theory. The indefinite introduces a discourse referent at the conceptual level of the DRS, which is then accessible to later pronouns. And the truth definition, specifying that the resulting DRS is true if and only if there is a way of embedding it in a model, determines that the discourse is true if and only if there is an animal, specifically a man, who is running.

Other kinds of determiners receive a different treatment. Let's turn to Burley's donkey sentence, 'Every farmer who owns a donkey beats it.' That is equivalent to 'If a farmer owns a donkey, he beats it.' We already know how to understand the antecedent; processing it yields the DRS $\langle u, v : \text{farmer}(u), \text{donkey}(v), u \text{ owns } v \rangle$. A DRS for a conditional has the form $m \Rightarrow m'$, where m and m' are DRSs for the antecedent and consequent. So, the DRS for this sentence becomes $\langle u, v : \text{farmer}(u), \text{donkey}(v), u \text{ owns } v \rangle \Rightarrow \langle w, x : w \text{ beats } x, w = u, x = v \rangle$. This is the general strategy for 'every,' which introduces a conditional structure. Notice that the phrase 'farmer who owns a donkey' now corresponds to an identifiable part of the semantic representation; there is no reason to adhere to the Fregean principle that only in the context of a sentence does an expression have meaning, though it is of course true that only in the context of a discourse does a sentence have specific truth conditions.

The semantic condition for conditionals makes it clear why 'every' lives on its subject term. A conditional $m \Rightarrow m'$ is true in a model \mathfrak{M} if and only if every embedding of m into \mathfrak{M} extends to an embedding of m' into \mathfrak{M} . This implies that the donkey sentence is true in \mathfrak{M} if and only if every submodel of \mathfrak{M} in which a farmer owns a donkey extends to one in which that farmer beats that donkey. But that is just to say that every farmer-owns-donkey pair in \mathfrak{M} is also a farmer-beats-donkey pair, just as we would expect.

Discourse Representation Theory, by adopting a dynamic strategy in which the meaning of a sentence is a function from discourse contexts to discourse contexts—or, viewed differently, a function from representations to representations, or, from still another point of view, from partial models to partial models—respects surface structure, derives truth conditions compositionally in rule-governed ways, explains anaphoric connections within sentences and across sentence boundaries, and assigns appropriate truth conditions. It does so, however, by treating NPs with different determiners differently. Some, such as 'a(n)' and 'some,' introduce discourse referents; their quantificational force arises from the semantics, but receives no direct representation. Others, such as 'every,' introduce conditionals but also receive no direct representation. Yet others, such as 'no,' 'never,' and so on, introduce negations. And some, such as 'many,' 'at least n ,' and 'uncountably many,' introduce plural discourse referents and conditions on them.

The theory of generalized quantifiers and Discourse Representation Theory thus pull in contrary directions. The former seeks a highly abstract theory encompassing all quantifiers and giving them a unified treatment. Discourse Representation Theory treats quantifiers of different kinds as performing very different kinds of tasks. It explains anaphoric behavior that the theory of generalized quantifiers does not address. In the process, however, it views many expressions traditionally viewed as quantificational as doing something else, and having quantificational force only by virtue of the truth conditions governing representations that themselves include nothing explicitly

quantificational. Attempts to combine these approaches are underway—see, for example, Barwise 1987, Kamp and Reyle 1993, Muskens 1996, van Eijck and Kamp 1997, Kamp, Genabith, and Reyle 2011—but remain in their infancy.

The task facing contemporary logicians, then, is to give a fully general and correct of the truth conditions of quantified sentences that explains their anaphoric behavior. It may seem discouraging that we face a situation startlingly like that facing Burley and Buridan in the middle of the fourteenth century—motivated, in fact, by many of the same linguistic examples. But we have, at least, a much richer set of tools with which to attempt the task.

6 References

Abbreviatio Montana. In N. Kretzmann and E. Stump (ed.), *The Cambridge Translations of Medieval Philosophical Texts, Volume One, Logic and the Philosophy of Language*. Cambridge: Cambridge University Press, 1988, 40–78.

Abbreviation Montana. In De Rijk 1967, 73–107.

Abelard, Peter, 1956. *Dialectica*. L.M. de Rijk (ed.), Assen: Van Gorcum.

Alexander of Aphrodisias, 1883.

. *Alexandri in Aristotelis Analyticorum Priorum librum I commentarium*, M. Wallies (ed.). Berolini: G.Reimeri.

Ariel, M., 2003. “Does *most* mean ‘more than half’?” *Berkeley Linguistics Society* 29: 17–30.

Ariel, M., 2004. “Most,” *Language* 80: 658–706.

Ariel, M., 2006. “A ‘just that’ lexical meaning for *most*,” in Klaus von Heusinger and Ken Turner (ed.), *Where semantics meets pragmatics*. Amsterdam: Elsevier, 49–91.

Aristotle. *De Interpretatione*. Translated by J. L. Ackrill. In J. Barnes (ed.), *The Complete Works of Aristotle*, Volume I, 25–38. Princeton: Princeton University Press, 1984.

Aristotle. *Posterior Analytics*. Translated by J. Barnes. In J. Barnes (ed.), *The Complete Works of Aristotle*, Volume I, 114–166. Princeton: Princeton University Press, 1984.

Aristotle. *Prior Analytics*. Translated by A. J. Jenkinson. In J. Barnes (ed.), *The Complete Works of Aristotle*, Volume I, 39–113. Princeton: Princeton University Press, 1984.

Aristotle. *Sophistical Refutations*. Translated by W. A. Pickard-Cambridge. In J. Barnes (ed.), *The Complete Works of Aristotle*, Volume I, 278–314. Princeton: Princeton University Press, 1984.

Arnauld, Antoine, and Nicole, Pierre, 1662. *La Logique: Ou, L'art de Penser (Logique de Port-Royal)*. Paris: Charles Savreux.

Arnauld, Antoine, and Nicole, Pierre, 1861. *Logic, Or, The Art of Thinking (The Port-Royal Logic)*. Translated by T. S. Baynes. Edinburgh: James Gordon.

Ars Burana. In De Rijk 1967, 175–213.

Ars Emmerana. In De Rijk 1967, 143–174.

Barnes, Jonathan, 1981. “Proof and the Syllogism,” in E. Berti, *Aristotle on Science: the 'Posterior Analytics'*. Padua: Antenore, 17–59.

Barwise, J. 1979. “On branching quantifiers in English,” *Journal of Philosophical Logic* 8: 47–80.

Barwise, J. and Cooper, R., 1981. “Generalized quantifiers and natural language,” *Linguistics and Philosophy* 4: 159–219.

Barwise, J. and Feferman, S. (eds.), 1985. *Model-Theoretic Logics* Berlin: Springer-Verlag.

Barwise, J., 1987. “Noun phrases, generalized quantifiers and anaphora,” in P. Gärdenfors (ed.), *Generalized Quantifiers: Linguistic and Logical Approaches*. Dordrecht: Reidel, 1–30.

Bentham, G., 1827. *Outline of a New System of Logic*. London: Hunt and Clarke.

Bochenski, I. M., 1961. *A History of Formal Logic*. Notre Dame: Notre Dame Press.

Boger, George, 2004. “Aristotle’s Underlying Logic,” In *Handbook of the History of Logic* 1, D. Gabbay and J. Woods (ed.). Amsterdam: Elsevier, 101–246.

Boole, G., 1847. *The Mathematical Analysis of Logic, being an Essay towards a Calculus of Deductive Reasoning*. Cambridge: Macmillan, Barclay, and Macmillan.

Boole, G., 1848. “The Calculus of Logic,” *Cambridge and Dublin Mathematical Journal*, 3: 183–198.

Boole, G., 1854. *An Investigation of the Laws of Thought, on Which are Founded the Mathematical Theories of Logic and Probabilities*. London: Walton and Maberly.

Buridan, John, 1976. *Tractatus de consequentiis*. H. Hubien (ed.), *Philosophes médiévaux* 16, Louvain: Publications Universitaires.

Buridan, John, 2001. *Summulae de Dialectica*. An annotated translation with a philosophical introduction by Gyula Klima. New Haven: Yale University Press.

Carlson, G., 1977. *Reference to Kinds in English*. PhD Dissertation, University of Massachusetts at Amherst.

Carroll, Lewis, 1896. *Symbolic Logic*. London: Macmillan.

Carroll, Lewis, 1977. *Lewis Carroll's Symbolic Logic*. New York: Potter.

Church, A., 1936. "A Note on the *Entscheidungsproblem*," *Journal of Symbolic Logic* 1: 40–41.

Corcoran, J., 1974, "Aristotle's Natural Deduction System," in J. Corcoran (ed.), *Ancient Logic and its Modern Interpretations*. Dordrecht: Reidel.

Corcoran, John, 1972. "Completeness of an Ancient Logic," *Journal of Symbolic Logic* 37: 696–705.

Corcoran, John, 1973. "A Mathematical Model of Aristotle's Syllogistic," *Archiv für Geschichte der Philosophie* 55: 191–219.

Davidson, D., 1967. "Truth and Meaning," *Synthese*, 17: 304–323.

de Rijk, L. M., 1967. *Logica Modernorum. A Contribution to the History of Early Terminist Logic. Vol. II*. Assen: Van Gorcum.

de Rijk, L. M., 1968. "On the Genuine Text of Peter of Spain's *Summulae logicales* I. General problems concerning possible interpolations in the manuscripts," *Vivarium* 6: 1–34.

de Rijk, L. M., 1982. "The Origins of the Theory of the Properties of Terms," *The Cambridge History of Later Medieval Philosophy*, edited by Norman Kretzmann, Anthony Kenny and Jan Pinborg. Cambridge: Cambridge University Press, 161–173.

DeMorgan, A., 1847. *Formal Logic or The Calculus of Inference*. London: Taylor and Walton.

DeMorgan, A., 1860. *Syllabus of a Proposed System of Logic*. London: Walton and Malbery.

Dutilh-Novaes, C., 2008, “Logic in the 14th Century after Ockham,” in D. Gabbay and J. Woods (eds.), *Handbook of the History of Logic, vol. 2, Medieval and Renaissance Logic*. Amsterdam: North Holland.

Frege, G., 1879. *Begriffsschrift*. Reprinted in van Heijenoort 1967, 1–82.

Frege, G., 1882. “Über die wissenschaftliche Berechtigung einer Begriffsschrift,” *Zeitschrift für Philosophie und Philosophische Kritik*, 81: 48–56.

Frege, G., 1892. “On Concept and Object,” originally published as “Über Begriff und Gegenstand,” in *Vierteljahresschrift für wissenschaftliche Philosophie* 16, 192–205. Translated in Geach, P., and Black, M., *Translations from the Philosophical writings of Gottlob Frege*, Oxford: Oxford University Press, 1952.

Frege, G., 1893. *Grundgesetze der Arithmetik*. Jena: Verlag Hermann Pohle.

Frege, G., 1902. “Letter to Russell,” in van Heijenoort 1967, 126–128.

Gödel, K., 1930. “The Completeness of the Axioms of the Functional Calculus of Logic,” in van Heijenoort 1967, 582–591.

Gödel, K., 1931. “On Formally Undecidable Propositions of *Principia Mathematica* and Related Systems I,” in van Heijenoort 1967, 596–616.

Geach, P., 1962. *Reference and Generality*. Ithaca: Cornell University Press.

Goldfarb, W., 1979. “Logic in the Twenties: the Nature of the Quantifier,” *Journal of Symbolic Logic* 44: 351–368.

Hackl, M., 2009. “On the grammar and processing of proportional quantifiers: *most* versus *more than half*,” *Natural Language Semantics* 17: 63–98.

Hailperin, T., 2004. “Algebraical Logic 1685–1900,” in D. Gabbay and J. Woods (ed.), *Handbook of the History of Logic*, volume 3. Amsterdam: Elsevier, 323–388.

Hamilton, William, 1860. *Lectures on Metaphysics and Logic*. Edinburgh and London: William Blackwood and Sons.

Heim, I., 1982. *The Semantics of Definite and Indefinite Noun Phrases*. Ph.D. thesis, University of Massachusetts, Amherst.

Henkin, L., 1961. “Some remarks on infinitely long formulas,” in *Infinitistic Methods*. Oxford: Pergamon Press, 167–183.

Heytesbury, William, 1335. *Regulae Solvendi Sophismata*. Pavia: Antonius de Carcano, 1481.

Hilpinen, R., 2004. "Peirce's Logic," in D. Gabbay and J. Woods (ed.), *Handbook of the History of Logic*, volume 3. Amsterdam: Elsevier, 611–658.

Hintikka, J., 1979. "Quantifiers vs. quantification theory," *Linguistics and Philosophy* 5: 49–79.

Introductiones Montana Minores. In De Rijk 1967, 7–71.

Irwin, Terence, 1988. *Aristotle's First Principles*. Oxford: Clarendon Press.

Jungius, Joachim, 1638. *Logica Hamburgensis*. R. Meyer (ed.). Hamburg: J. J. Augustin, 1957.

Jungius, Joachim, 1977. *Logica Hamburgensis Additamenta*. Edited by Wilhelm Risse. Göttingen: Vandenhoeck and Ruprecht.

John Philoponus, 1905. *In Aristotelis Analytica Priora commentaria*. M. Wallies, ed. Berolini: G.Reimeri.

Johnson, Fred, 1994. "Apodictic Syllogisms: Deductions and Decision Procedures," *History and Philosophy of Logic* 16: 1–18.

Kamp, H., 1981. "A theory of truth and semantic representation," in J. A. G. Groenendijk, T. M. V. Janssen, and M. B. J. Stokhof (eds.), *Formal Methods in the Study of Language*. Amsterdam: Mathematical Centre Tracts 135: 277–322.

Kamp, H., and Reyle, U., 1993. *From Discourse to Logic*. Dordrecht: Kluwer.

Kamp, H., Genabith J., and U. Reyle, 2011. "Discourse Representation Theory," in D. Gabbay (ed.), *Handbook of Philosophical Logic* (second edition), Springer.

Karttunen, L., 1976. "Discourse Referents," in J. D. McCawley (ed.), *Syntax and Semantics 7: Notes from the Linguistic Underground*. New York: Academic Press, 363–385.

Keenan, Edward L., and Moss, L., 1984. "Generalized quantifiers and the expressive power of natural language," in J. van Benthem and A. ter Meulen (eds.), *Generalized quantifiers in Natural Language*. Dordrecht: Foris, 73–124.

Keenan, Edward L., and Stavi, J., 1986. "A Semantic Characterization of Natural Language Determiners," *Linguistics and Philosophy* 9: 253–326.

Keenan, E., 1987. "A semantic definition of 'Indefinite NP'," in Eric Reuland and Alice ter Meulen (ed.), *The Representation of (In)definiteness*. Cambridge: MIT Press,

286–317.

Keenan, Edward L., 1996. “The Semantics of Determiners,” in Shalom Lappin (ed.), *The Handbook of Contemporary Semantic Theory*. Oxford: Blackwell.

Keenan, Edward L., and Westerståhl, D., 1997. “Generalized Quantifiers in Linguistics and Logic,” in *Handbook of Logic and Language*. Amsterdam: Elsevier, 837–893.

Keenan, E., and Westerståhl, D., 2011. “Generalized quantifiers in Linguistics and Logic (revised version),” in J. van Benthem and A. ter Meulen (eds.), *Handbook of Logic and Language*. Second Edition, Amsterdam: Elsevier, 859–910.

King, P., 1985. *Jean Buridan’s Logic. The Treatise on Supposition. The Treatise on Consequences*. Translation with introduction and notes. Synthese Historical Library 27, Dordrecht: Reidel.

Kneale, William and Kneale, Martha, 1962. *The Development of Logic* Oxford: Oxford University Press.

Kretzmann, Norman, 1966. *William of Sherwood’s Logic*. Minneapolis: University of Minnesota Press.

Lear, Jonathan, 1980. *Aristotle and Logical Theory*. Cambridge: Cambridge University Press.

Lewis, D., 1975. “Adverbs of quantification,” in E. Keenan (ed.), *Formal Semantics of Natural Language*. Cambridge: Cambridge University Press, 3–15.

Lindström, P., 1966. “First Order Predicate Logic with Generalized Quantifiers,” *Theoria* 32: 186–195.

Lindström, P., 1969. “On Extensions of Elementary Logic,” *Theoria* 35: 1–11.

Lindström, P., 1974. “On Characterizing Elementary Logic,” in S. Stenlund (ed.), *Logical theory and semantic analysis*. Dordrecht: D. Reidel, 129–146.

Löwenheim, L., 1915. “On Possibilities in the Calculus of Relatives,” in van Heijenoort 1967, 228–251.

Lukasiewicz, J., 1934. “Outlines of the History of the Propositional Logic,” *Pr. Fil.* 37.

Lukasiewicz, Jan, 1951. *Aristotle’s Syllogistic from the Standpoint of Modern Formal Logic*. Oxford: Clarendon Press.

Martin, C., 2009, “The logical text-books and their influence,” in J. Marenbon (ed.), *The Cambridge Companion to Boethius*. Cambridge: Cambridge University Press.

McKiraahan, Richard, 1992. *Principles and Proofs*. Princeton: Princeton University Press.

Montague, R., 1973. “The Proper Treatment of Quantification in Ordinary English,” in J. Hintikka, J. Moravcsik, and P. Suppes (ed.), *Approaches to Natural Language*. Dordrecht: D. Reidel, 1973, 221–242. Reprinted in Montague 1974, 247–270.

Montague, R., 1974. *Formal Philosophy* (edited and with an introduction by R. Thomason). Yale University Press, New Haven.

Mostowski, A., 1957. “On a generalization of quantifiers,” *Fundamenta Mathematicae*, 44: 12–36.

Muskens, R., 1996. “Combining Montague Semantics and Discourse Representation,” *Linguistics and Philosophy* 19: 143–186.

Normore, C., 1999, “Some Aspects of Ockham’s Logic,” in P. V. Spade (ed.), *The Cambridge Companion to Ockham*. Cambridge: Cambridge University Press.

Ockham, William of, 1974. *Summa Logicae*. Ph. Boehner, G. Gál and S. Brown (eds.), *Opera Philosophica I*, St. Bonaventure, NY: The Franciscan Institute.

Ockham, William of, 1974. *Ockham’s Theory of Terms: Part I of the Summa Logicae*. Notre Dame: Notre Dame Press.

Ockham, William of, 1980. *Ockham’s Theory of Propositions: Part II of the Summa Logicae*. Notre Dame: Notre Dame Press.

Patzig, Günther, 1969. *Aristotle’s Theory of the Syllogism*. Translation, Jonathan Barnes. Dordrecht: D. Reidel.

Peano, G., 1889. “The Principles of Arithmetic, Presented by a New Method,” in van Heijenoort 1967, 83–97.

Peirce, C. S., 1885, “On the Algebra of Logic: A Contribution to the Philosophy of Notation,” *American Journal of Mathematics* 7 (1885), 180–196.

Peirce, C. S., 1893, *Elements of Logic, Collected Papers of Charles Sanders Peirce*, Volume II, edited by C. Hartshorne and P. Weiss, Cambridge: Belknap Press, 1960.

Peter of Spain. *Synkategoreumata*. First Critical Edition with an Introduction and Indexes by L.M. de Rijk, with an English Translation by Joke Spruyt. Leiden/Kln/New

York, 1992.

Peter of Spain. *Tractatus called afterwards Summule logicales*. First Critical Edition from the Manuscripts with an Introduction by L.M. de Rijk. Assen, 1972.

Peters, S. and westerståhl, D., 2006. *Quantifiers in Language and Logic*. Oxford: Oxford University Press.

Pietroski, P., Lidz, J., Hunter, T., and Halberda, J., 2009. "The meaning of 'most': Semantics, numerosity and psychology," *Mind and Language* 24: 554–585.

Quine, W. V. O., 1939. "A Logistical Approach to the Ontological Problem," Fifth International Congress for the Unity of Science, Cambridge, Mass., September 9, 1939. Reprinted in *The Ways of Paradox*. Cambridge: Harvard University Press, 1966.

Quine, W. V. O., 1956. "Quantifiers and Propositional Attitudes," *The Journal of Philosophy* 53: 177–187. Reprinted in *The Ways of Paradox*. Cambridge: Harvard University Press.

Quine, W. V. O., 1960. *Word and Object*. Cambridge: MIT Press.

Rescher, Nicholas, 1964. "Plurality Quantification," *The Journal of Symbolic Logic* 27 (1964): 373–74.

Ross, W. D. (ed.), 1951. *Aristotle's Prior and Posterior Analytics*. Oxford: Clarendon Press.

Russell, B., 1902. "Letter to Frege," in van Heijenoort 1967, 124–125.

Sher, G. 1997. "Partially-ordered (branching) generalized quantifiers: a general definition," *Journal of Philosophical Logic* 26: 1–43.

Skolem, T., 1920. "Logico-combinatorial Investigations in the Satisfiability or Provability of Mathematical Propositions: A Simplified Proof of a Theorem by L. Löwenheim and Generalizations of the Theorem," in van Heijenoort 1967, 252–263.

Skolem, T., 1928. "On Mathematical Logic," in van Heijenoort 1967, 508–524.

Smiley, Timothy, 1974. "What Is a Syllogism?" *Journal of Philosophical Logic* 1: 136–154.

Smiley, Timothy, 1994. "Aristotle's Completeness Proof," *Ancient Philosophy* 14: 25–38.

Smith, Robin, 1989. *Aristotle's Prior Analytics*. Indianapolis: Hackett.

Spade, P. V., 2002. *Thoughts, Words, and Things: An Introduction to Medieval Logic and Semantic Theory*.

Striker, Gisela (tr.), 2009. *Aristotle, Prior Analytics I*. Translated with commentary by Gisela Striker. Oxford: Clarendon Press.

Tarski, A., 1936. "Über den Begriff der logischen Folgerung," *Actes du Congrès international de philosophie scientifique*, Sorbonne, Paris 1935, vol. VII, Logique, Paris: Hermann, 1–11. Reprinted as "The Concept of Truth in Formalized Languages" in *Logic, Semantics, Metamathematics: Papers from 1923 to 1938*. Edited and translated by J. H. Woodger. Oxford: Oxford University Press, 1956.

Tarski, A., and Vaught, R., 1956. "Arithmetical Extensions of Relational Systems," *Compositio Mathematica* 13: 81–102.

Tractatus Anagnini. In De Rijk 1967, 215–332.

Valencia, V. S., 2004. "The Algebra of Logic," in D. Gabbay and J. Woods (ed.), *Handbook of the History of Logic*, volume 3. Amsterdam: Elsevier, 389–544.

van Benthem, J., 1983. "Determiners and Logic," *Linguistics and Philosophy* 6: 447–478.

van Benthem, J., 1984. "Questions about Quantifiers," *Journal of Symbolic Logic* 49: 443–466.

van Benthem, J., 1986. *Essays in Logical Semantics*. Dordrecht: D. Reidel.

van Benthem, J., 1987. "Towards a computational semantics," in P. Gärdenfors (ed.), *Generalized Quantifiers*, Dordrecht: D. Reidel, 31–71.

van Benthem, J., 1989. "Polyadic Quantifiers," *Linguistics and Philosophy* 12: 437–464.

van der Does, J., and van Eijck, J., 1996. "Basic Quantifier Theory," in J. van der Does and J. van Eijck (ed.), *Quantifiers, Logic, and Language*. Stanford: CSLI.

van Eijck, J., and Kamp, H., 1997. "Representing discourse in context," in J. van Benthem and A. ter Meulen (eds.), *Handbook of Logic and Language*. Amsterdam: Elsevier, 179–237.

van Heijenoort, J., 1967. *From Frege to Gödel: A Source Book in Mathematical Logic 1879–1931*. Cambridge: Harvard University Press.

Wallace, J., 1970. "On the Frame of Reference," *Synthese* 22: 117–150.

Watts, Isaac, 1725. *Logic, or The Right Use of Reason in the Enquiry After Truth With a Variety of Rules to Guard Against Error in the Affairs of Religion and Human Life, as well as in the Sciences*. Edinburgh: Printed for Charles Elliot.

Wedin, Michael V., 1990. "Negation and Quantification in Aristotle," *History and Philosophy of Logic* 11: 131–150.

Westerståhl, D., 1987. "Branching generalized quantifiers and natural language," in P. Gärdenfors (ed.), *Generalized Quantifiers*. Dordrecht: D. Reidel, 269–298.

Westerståhl, D., 1989, "Quantifiers in Formal and Natural Languages," in D. Gabbay and F. Guentner (eds), *Handbook of Philosophical Logic*, Vol. IV, Dordrecht: D. Reidel, 1–131. 2nd edition, 2007, Berlin: Springer, 223–338.

Whately, Richard, 1826. *The Elements of Logic*. London: Longmans.

Wilson, C., 1960. *William Heytesbury: Medieval Logic and the Rise of Mathematical Physics*. Madison: University of Wisconsin Press.

Woods, J., and Irvine, A., 2004. "Aristotle's Early Logic," in D. Gabbay and J. Woods (ed.), *Handbook of the History of Logic*, volume 1. Amsterdam: Elsevier, 27–100.