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FREEDOM AND TRUTH IN MATHEMATICS

"The very essence of mathematics," Georg Cantor wrote in 1883, "is its freedom."[1] Few philosophers of mathematics in the intervening 100 years have taken Cantor's view seriously. The paradoxes of set theory struck a seemingly fatal blow to Cantor's libertarian attitude toward mathematics; the chief task of work in the foundations of mathematics became the restriction of mathematical activity to safer realms. Constructivists, to take just one example, have sought to tighten the criteria for asserting the existence of mathematical objects.Whatever their approach, philosophers of mathematics have attempted to develop an epistemological account of these objects, which, in light of the paradoxes, have seemed more obscure and puzzling than ever.

In this paper I shall sketch a theory of mathematical truth that runs counter to these trends by taking Cantor's attitude as an inspiration. I am advocating that we grant mathematicians all the liberty that the avoidance of contradiction allows. My policy, I shall argue, permits us to explain many commonly held views toward mathematics: (1) that mathematical statements are either necessarily true or necessarily false; (2) that mathematical truth derives ultimately from logical truth; and (3) that existence in mathematics involves a sort of modality, requiring only consistency or constructibility. In addition, my theory casts results concerning consistency and incompleteness as well as proposals for the extension of mathematical systems in a radically new light.

I

I shall present my theory of truth in mathematics as a semantics for mathematical language, specifically, for the language of Zermelo-Fraenkel set theory. I shall assume that we have a countable set of individual variables, represented metalinguistically by 'x', 'x_1' etc., the binary predicate constants '$\in$'and '$=$', and the logical symbols '$\exists$', '&', '$-$', '(', and ')'.[2]

For definitional convenience, consider the extension L' of this lan-

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guage L which contains, in addition, a countable set of individual constants represented metalinguistically by 'a', 'a_1', etc. A (binary) predicate of L' flanked by two constants is an L'-formula; is A and B are L'-formulas, so are -A and (A & B). If A is an L'-formula with a constant a, and to which the variable x is foreign, then $(\exists x)Ax/a$ (the result of prefixing $(\exists x)$ and substituting x for all occurences of a throughout A) is an L'-formula too. The set of L'-formulas is the smallest satisfying these conditions. An L-formula is an L'-formula with no individual constants.

The semantics I am advocating generalizes both referential and substitutional approaches. To simplify, I shall count an existentially quantified sentence true if and only if it has a true instance in a parametric extension of the language that meets certain criteria. Where L is any first-order language, a *parametric model* M is an ordered pair $(\propto, \Gamma)$, where $\propto$ (M's *assignment function*) is a mapping from atomic formulas of L into truth values and Γ (M's *game set*) is a set of mappings β from atomic formulas of parametric extensions of L into truth values such that (a) $\propto \in \Gamma$ and (b) for each β, β restricted to the constants of L is identical to $\propto$ (symbolically, $\beta \upharpoonright L = \propto$). I shall say that a parametric model M' (in language L'), M' = $(\propto', \Gamma')$, *parametrically extends* another (in language L), M = $(\propto, \Gamma)$, just in case (a) $\beta' \subsetneq \Gamma$ and (b) every β such that $\beta \in \Gamma$ and $\beta \upharpoonright L' = \propto'$ is such that $\beta \in \Gamma'$. It follows, of course, that $\propto' \upharpoonright L = \propto$, that $\propto' \in \Gamma$, and that L' adds at most some individual constants to L.

The general idea behind these definitions is that M's game set specifies the class of admissible extensions of the assignment $\propto$, i.e., the class of all extensions that conform to the operative language game. M' extends M only if $\propto'$ is an admissible extension of $\propto$ $(\propto' \in \Gamma)$ and only if M''s game set includes just those admissible extensions of $\propto$ that also extend $\propto'$. M' thus extends M in accordance with the language game that Γ specifies.

I shall define the valuation function on a model M, for the language of ZF, recursively :

1. $V_M(a_1Fa_2) = T$ iff $\propto (a_1Fa_2) = T$.

2. $V_M(-A) = T$ iff $V_M(A) = F$.

3. $V_M((A \& B)) = T$ iff $V_M(A) = V_M(B) = T$.

4. $V_M((\exists x)A) = T$ iff $V_{M'}(Ac/x) = T$
for some constant c and parametric extension M' of M.

The language of ZF as I am construing it has no individual constants. Any parametric model of ZF must therefore have a null assignment function; the properties of a model will depend entirely on its game set. How can we characterize an apropriate game set?

II

I intend the term 'game set' to suggest an analogy with game-theoretic semantics.[3] The game set Γ of a parametic model M = (∝,Γ) is the only feature distinguishing M from a standard substitutional model. Game sets make parametric semantics more powerful than typical substitutional approaches by allowing us both to extend the language and to constrain the class of admissible extensions. 'There is a pebble in my shoe' can be true whether or not our language contains a name for the pebble, so long as we can extend our language to include such a name. We must restrict game sets so that such sentences are true just in the appropriate circumstances, e.g., in those cases where there really is a pebble in my shoe. Often our criteria for restriction will be referential. But we can allow that 'Some flying horse is named Pegasus' varies in truth value with the operative game set, coming out true on mythology and false on biology examinations. Game sets will reflect and formally represent the context of utterance.

How can we characterize mathematical contexts? How, that is, should we restrict game sets when we interpret a mathematical theory such as ZF? We might insist that Γ include only extensions that justify the existentials that follow from axioms or that assert the existence of entities we can give effective procedures for constructing. Taking Cantor's libertarian attitude as an inspiration, we might decide to countenance *all* extensions, placing no restrictions on the game sets.

If we adopt this approach, the most liberal possible toward the admissibility of parametric extensions, we obtain the *semantics of open models.*

Say that a parametric model $M = (\propto, \Gamma)$ is *open* just in case every β such that $\beta \upharpoonright L = \propto$ is in Γ. Then call a formula A *OM-valid* just in case $V_M(A) = T$, for every open model M, *OM-contradictory* just in case $V_M(A) = F$, for every open model M, and *OM-contingent* otherwise.

The semantics of open models has some surprising properties. In particular we can show that every quantified sentence is either OM-valid or OM-contradictory.Since every formula of the language of ZF is quantified, it follows that every ZF formula is either OM-valid or OM-contradictory. Furthermore, we can show quite easily that the class of OM-valid formulas is not recursively enumerable; if we let ZF* be the set of OM-valid formulas in the language of ZF, then ZF* is not axiomatizable.[4]

Despite the suggestiveness of these properties, however, the semantics of open models does not suffice as an interpretation of ZF or other mathematical theories. ZF implies that no set has itself as an element, i.e., that $(\exists x)\ (x \in x)$ is false. Unfortunately, where $M = (\emptyset, \Gamma)$, there is a parametric extension M' of M such that $\propto'\ (a \in a) = T$, so $V_M((\exists x)\ (x \in x)) = T$. The semantics of open models makes every satisfiable existential formula true and, if we let (x) abbreviate $-(\exists x)-$, every falsifiable universal formula false. Since some theorems of ZF have initial universal quantifiers, yet are not logical truths, the semantics of open models fails to produce a valuation appropriate to set theory.

III

The semantics of open models counts an existential sentence true just in case it is consistent with logical principles. In so doing it counts too many such sentences true, including many that violate basic principles of set theory. To remedy this failing, I shall count an existential formula true if and only if it accords with logical and set-theoretic principles. In the general case, let Δ be a set of formulas which intuitively characterize the mathematical structures we intend the theory to describe. Normally we will restrict Δ to sentences with initial universal quantifiers so that Δ itself has no ontological commitments.[5] A parametric model $M = (\propto, \Gamma)$ *conforms to* Δ (or, *is* Δ *- conforming*) just in case, for any β, $\beta \in \Gamma$ if and only if (a) $\beta \upharpoonright L = \propto$ and (b) for each A

$\in \Delta$, $V_{M'}(A) = T$, where M' = (β, {β}). I shall call parametric semantics restricted to the class of Δ -conforming models the *semantics of Δ-conforming models.*

This semantics has much in common with the semantics of open models. We can show, for example, that any L-formula having the form $(\exists x)A$ is true on a Δ-conforming model just in case it is consistent with Δ; any L-formula having the form $(x)A$ is true on such a model if and only if it is implied by Δ. If we define Δ-validity and Δ-contradictoriness in the obvious way, we can show that any quantified formula – and, so, any formula in the language of set theory – is either Δ-valid or Δ-contradictory. Let ZF* be the set of Δ-valid formulas in the language of ZF; we can show that ZF* is not axiomatizable. The interesting features of the semantics of open models, therefore, remain when we tighten the reins to obtain the semantics of Δ-conforming models.

IV

The semantics of Δ-conforming models reflects a view of mathematics as the study of possible structures.[6] The set Δ characterizes the class of structures we wish to investigate; the set of Δ-valid sentences constitutes the theory of the structures that results from this choice. The basic insight, which is Cantor's, is that all a mathematical object needs to "do" to exist is to be possible from the perspective of Δ. We want to examine all possible structures of a certain kind; we have interest, therefore, in every structure compatible with the stipulations that rule out the irrelevant structures. Mathematicians should thus be free to speak of any object meeting the basic requirements and producing no inconsistencies. The distinction between actual and possible structures does not apply.

In the case of ZF, our choice of Δ permits a startling degree of simplicity. We want to include the axioms that are universal in form and appear fundamental to our concept of "set". We need to include in Δ, therefore, the axioms of extensionality, pair set, sum set, power set, foundations and (if we wish) choice.[7] Each arguably has a clear intuitive basis in our notion of the class of structures we intend to describe. Notice that we do not need a null set axiom; given our choice of Δ, $(\exists x)\,(y)\,(y \notin x)$ is already Δ-valid. The axiom of infinity is similarly superfluous. Far more significantly, note that we do not need axioms of subsets or replacement. The comprehension principle that led Cantor into the paradoxes is demonstrably false on this semantics, since it is not consistent with Δ. Nevertheless we can show that every instance of Cantor's comprehension axiom that is consistent with Δ is Δ-valid. We get the effect of axioms of subsets and replacement without independently assuming them. Our semantics thus destroys the chief obstacle to Cantor's "naive" approach to set theory.[8]

The semantics of Δ-conforming models also allows us to make sense of the traditional view that mathematical statements are necessarily true if true at all. We may think of necessary truth as truth on all relevant models; the set Δ picks out which models are relevant to assessments of mathematical truth. Since every formula in the language of set theory – or any other mathematical theory, so long as we exclude constants – is either Δ-valid or Δ-contradictory, we can give a precise sense to the assertion that all mathematical statements are either necessarily true or necessarily false.

Similarly, the semantics of Δ-conforming models accounts for the commonly held view that mathematical truth is in some sense parasitic on logical truth. We decide what truth value to give a quantified sentence by determining its logical relation to the set Δ. Once we have defined the notion of consistency and chosen an appropriate Δ, the semantics of Δ-conforming models characterizes truth in the mathematical theory specified by Δ. My semantics thus also explicates the view, held by David Hilbert with regard to «ideal» objects, for example, that mathematical existence amounts to nothing more than consistency.[9] To show that a mathematical object exists we need verify only that the assertion that it exists is consistent with the set of (nonexistential) formulas specifying the kind of mathematical structure we care about.

Paul Benacerraf argues that theories of mathematical truth face a dilemma: they must either make mathematical knowledge mysterious (by assuming entities with which we stand in no epistemic contact) or loose semantical accounts of mathematical discourse from the anchor (namely, reference) that links them to the rest of language.[10] From the perspective of the semantics of Δ-conforming models, we can see that mathematical knowledge – though rooted in rather vague intuitive notions and at times very difficult to attain, due to the complexities of the problem of consistency – requires no occult faculties. Neglecting for a moment the question of our choice of an appropriate Δ, we can explain mathematical knowledge if we can explain knowledge of logical relations such as consistency. Furthermore, we can see how innocuous the ontological commitment of mathematical theories to abstract entities is. The set Δ has no ontological commitments; the commitments of the total theory can be explained and, ultimately, explained away in terms of logic.[11] Of course, I have adopted a seemingly peculiar semantics for mathematical discourse. I insist, however, that parametric semantics can account for quantification in natural and formal languages, in mathematical and nonmathematical contexts. I cannot argue for such a sweeping claim here. But, if I am correct, the restriction on game sets that characterizes mathematical contexts has the same status as restrictions we might adopt for other realms of discourse, e.g., fictional or micro-theoretical realms.

V

I have urged that the semantics of Δ-conforming models provides a framework for an adequate philosophical theory of mathematical truth. I also contend that the semantics provides a satisfactory framework for understanding mathematical activity and practice.

First, consider problems of consistency. To introduce a set by way of the axiom of subsets, for example, a set theorist must characterize a background set and a consistent principle of selecting subsets of it. The axiom thus systematizes the burden incumbent on the mathematician of demonstrating consistency. A desire to introduce sets unattainable through other means but producing no inconsistencies clearly motivates the axiom of subsets. We may view the axiom, then, as an attempt to solve the problem of consistency that my semantics poses.

My semantics also makes Gödel's incompleteness results seem less surprising. We can show without using any metamathematical techniques beyond those of Church's theorem that ZF*, if consistent, is not recursively enumerable. Any existential formula consistent with Δ is true; no wonder, then, that not all truths follow from any consistent set of axioms.

Finally, the semantics of Δ-conforming models suggests a program for handling proposed extensions of mathematical systems. Some have suggested, for example, that we extend set theory by taking as axioms such "undecidable" sentences as ConZF, Con(ZF U {ConZF}), etc. But on my semantics all these formulas are already in ZF*. Other proposed axioms – asserting the existence of measurable or inaccessible cardinals – have the same status in their usual formulations. They are provably true on the semantics of Δ-conforming models. Their addition consequently merits little concern provided that they are consistent with Δ; we may add them without revising our intuitive notions. The semantics of Δ-conforming models thus does much to explain the attitude of contemporary set theorists that proposed additions to our theories are to be evaluated primarily on the basis of their consistency. In contrast, the continuum hypothesis (and the axiom of choice, if we omit it from Δ initially) comes out false on my semantics. This constitutes no argument against the hypothesis, but suggests that its adoption would require a revision of our basic concept of "set" in a way that adopting an axiom of inaccessible cardinals would not. We can say that the continuum hypothesis is false given a certain basic notion of "set"; the more pragmatic question remains. Should we incorporate the hypothesis into our basic conception? To this broadly methodological question, of course, my semantics provides no answer.

I have been contending that the semantics of Δ-conforming models constitutes the keystone of an adequate theory of mathematical truth. It explains how mathematical knowledge is possible, why mathematical truths are necessary, how mathematics depends on logic, and how we ought to analyse questions concerning the status and possible extension of mathematical systems. In the process it breathes new life into Cantor's set theory and into his libertarian attitude toward mathematical thinking. Much more must be said to develop and defend the account I am offering. If I am right, nevertheless, philosophical attempts to build walls around mathematicians have fundamentally misdirected the philosophy of mathematics for nearly a century.

NOTES

1. 'Über unendliche, lineare Punktmannigfaltigkeiten,' *Math. Annalen* **21** (1883), 545-591, p. 564; quoted in Abraham A. Fraekel, *Abstract Set Theory* (Amsterdam: North-Holland, 1976), p.2.

2. I am following Fraenkel and Bar-Hillel in considering '=' as defined within the theory; see *Foundations of Set Theory* (Amsterdam: North-Holland, 1958), pp. 28-33. In the general case, of course, we might have '=' as a logical symbol. We might also have finitely or countably many n-ary predicate constants, for each n, and a finite or countable set of individual constants. Throughout the remainder of the paper I shall assume that all our symbols name themselves.

3. For a basic discussion of game-theoretic semantics, see K.J.J. Hintikka, 'Language Games for Quantifiers,' in *Logic, Language Games and Information* (Oxford : Clarendon, 1973), pp. 53-82. For an intriguing analysis of Hintikka's work that bears on the link between game-theoretic and substitutional approaches see Robert Kraut, *Objects* (PhD dissertation, University of Pittsburgh, 1976), pp. 1-24.Kraut and I use 'game set' in different but related ways; see pp. 69-72.

4. For proofs of these and upcoming formal results, see my 'Parametric Substitutional Semantics,' forthcoming. As detailed there, these results require a slight revision of the recursion clauses involving quantifiers.

5. A thorough discussion of ontological commitment supporting this assertion inhabits my *Reduction in the Abstract Sciences* (Indianapolis: Hackett, 1982), chapter 8. My distinction between universal and existential axioms is similar to Ermanno Bencivenga's distinction between the logic of sets and unconditioned existence assertions. See 'Set Theory and Free Logic,' *Journal of Philosophical Logic* **5** (1976), 1-15, p. 13; 'Are Arithmetical Truths Analytic: New Results in Free Set Theory,' *Journal of Philosophical Logic* **6** (1977), 319-330, p. 319.

6. Hilary Putnam, 'Mathematics Without Foundations,' in *Mathematics Matter and Method* (Cambridge: Cambridge University Press, 1975), pp. 43-59, Michael Jubien, 'Ontology and Mathematical Truth,' *Noûs* **11** (1977), 133-149, and Philip Kitcher, 'The Plight of the Platonist,' *Noûs* **12** (1978), 119-136, offer views of mathematics as the study of possible structures of certain kinds. Few if any of the following advantages, however, accrue to their analyses.

7. For statements of these axioms, see Fraenkel and Bar-Hillel, pp. 33-37, 46, 91; Paul J. Cohen, *Set Theory and the Continuum Hypothesis* (Reading, Mass.: W.A. Benjamin, 1966), pp. 51-53; W.V.O. Quine, *Set Theory and Its Logic* (Cambridge, Mass.: Harvard, 1963), p. 331.

8. I am arguing not that we should "write off" 70 years of research in axiomatic set theory but that we should see axiomatic developments of set theory as providing incomplete syntactic approximations to the semantics I propose.

9. See 'On the Infinite', in Jean van Heijenoort (ed.), *From Frege to Gödel: A Source Book in Mathematical Logic, 1879-1931* (Cambridge, Mass.: Harvard, 1967), pp. 369-392, pp. 370, 383.

10. 'Mathematical Truth,' *Journal of Philosophy* **70** (1973), 661-679. For a discussion of Benacerraf's dilemma, see *Reduction in the Abstract Sciences,* chapter 1.

11. This, of course, is no easy task, but requires (at least) a nominalistic acount of logical validity.

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